On partial match queries in two-dimensional quadtrees.

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J.W. Goethe Universität Frankfurt a.M.

(joint work with N. Broutin and H. Sulzbach)

Analysis of Algorithms (2011),
Bedlewo, Poland

June 14, 2011

The (point) quadtree

Data from $\mathbb{R}^2$ (dimension 2)

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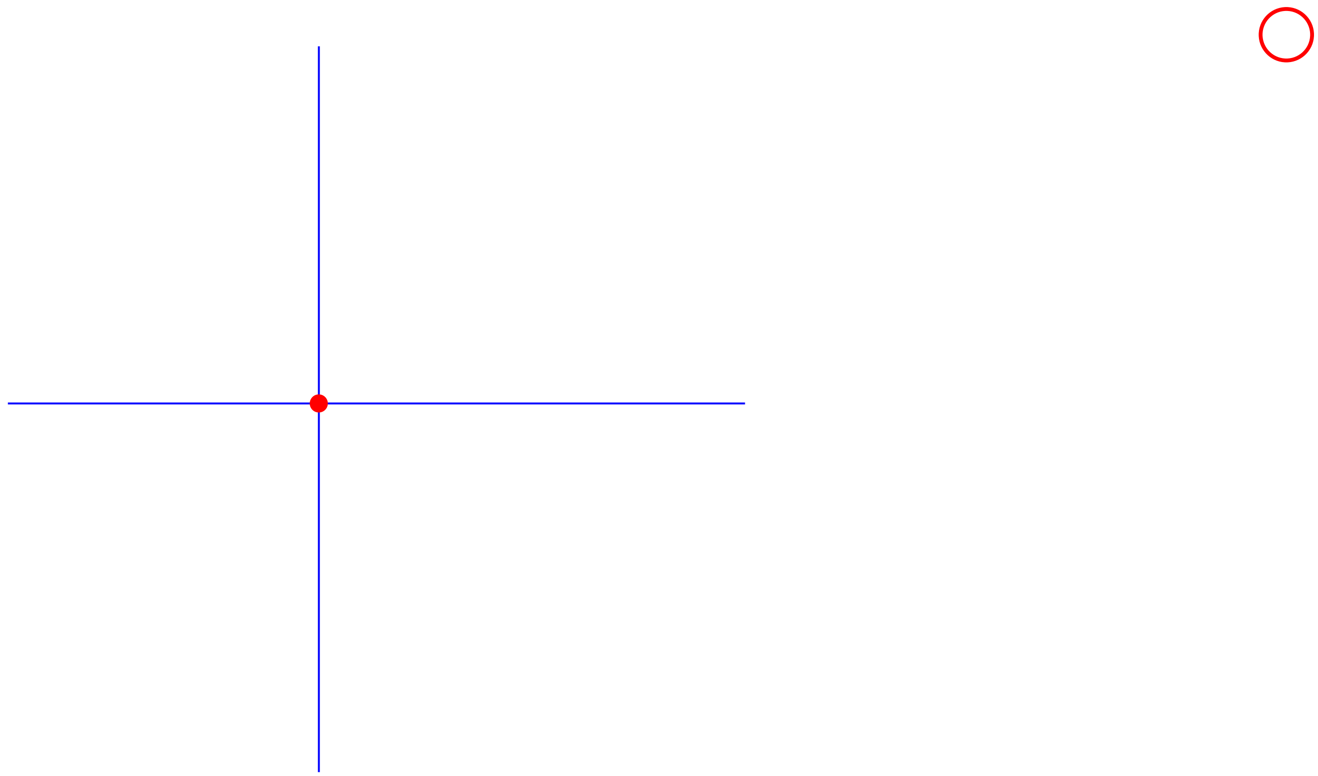

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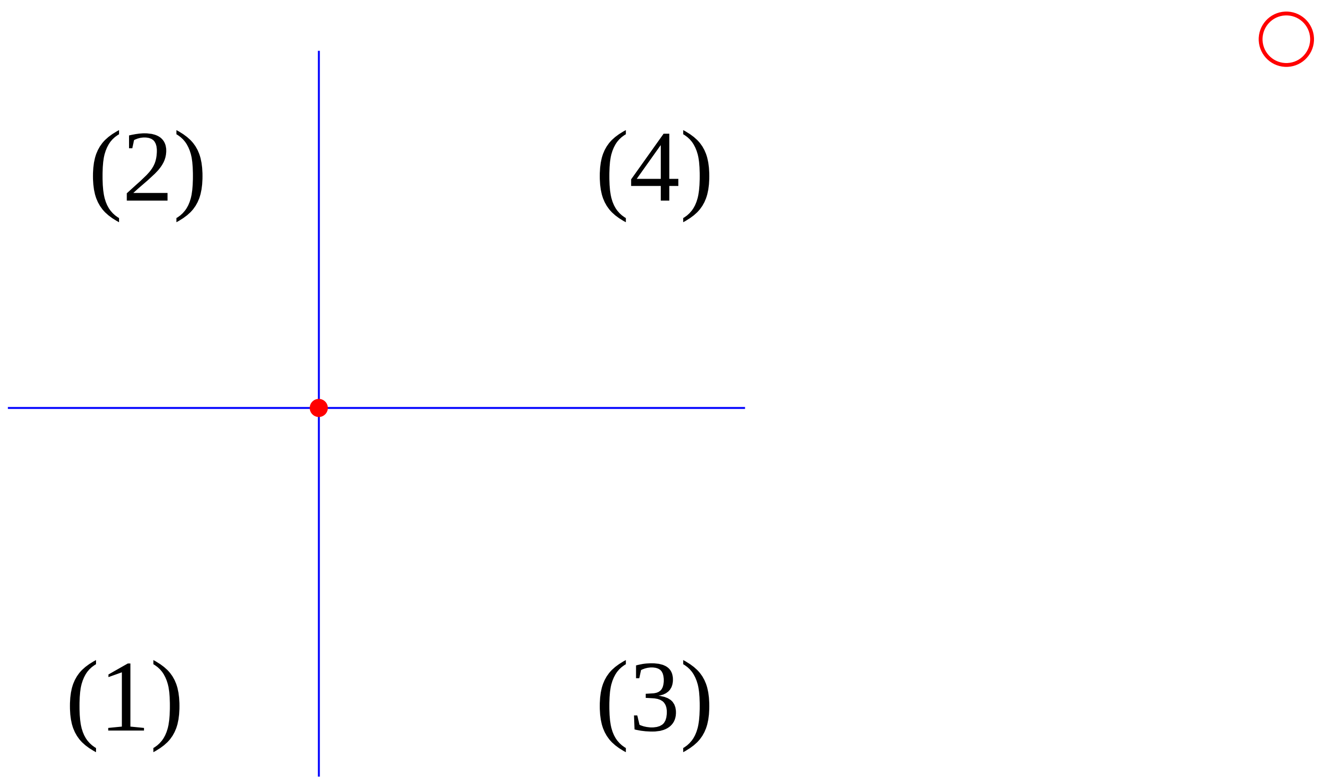

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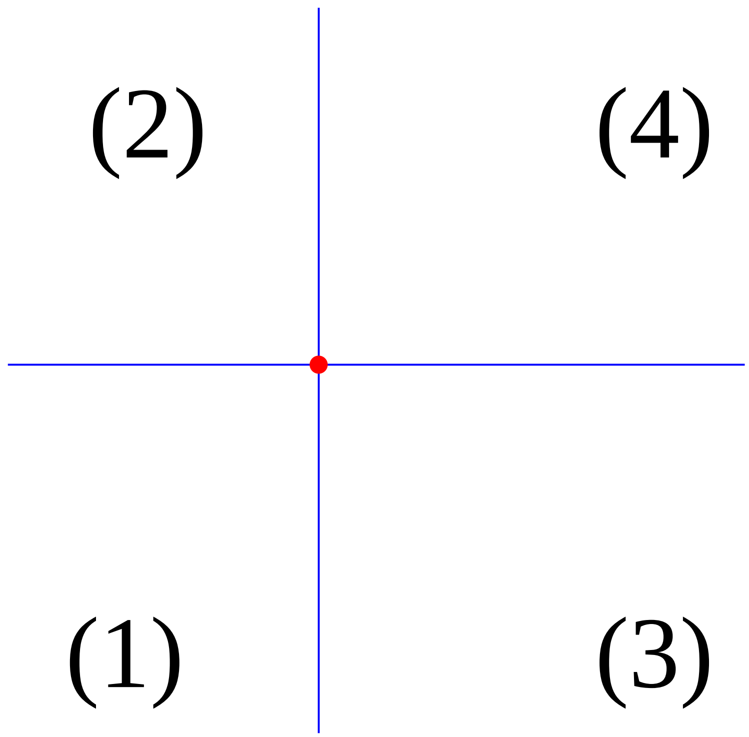

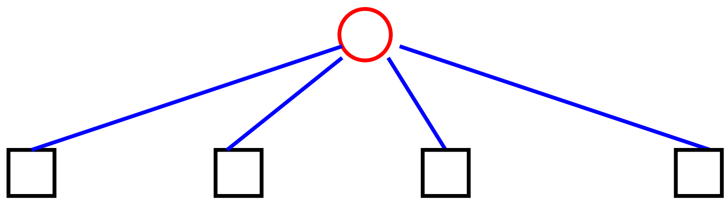

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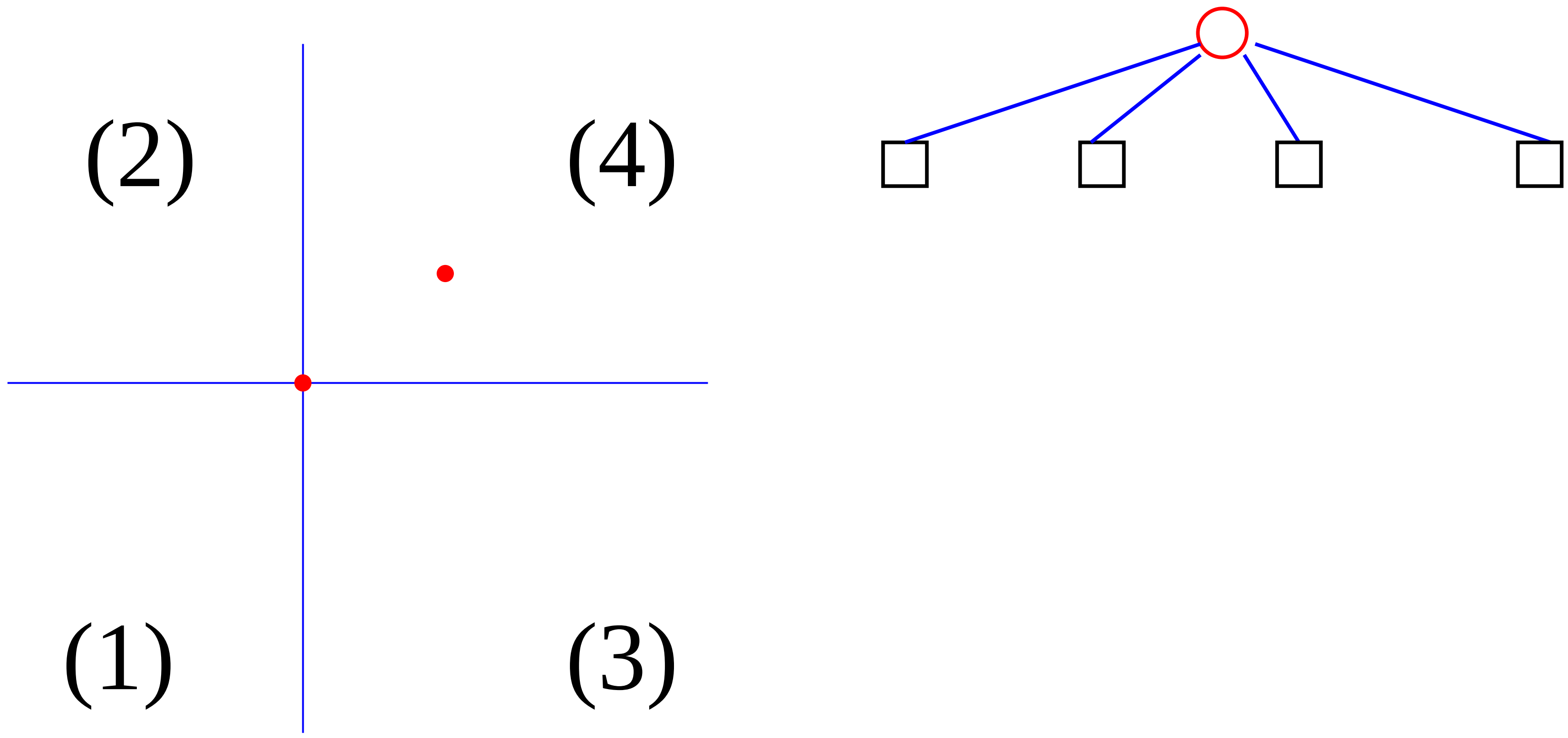

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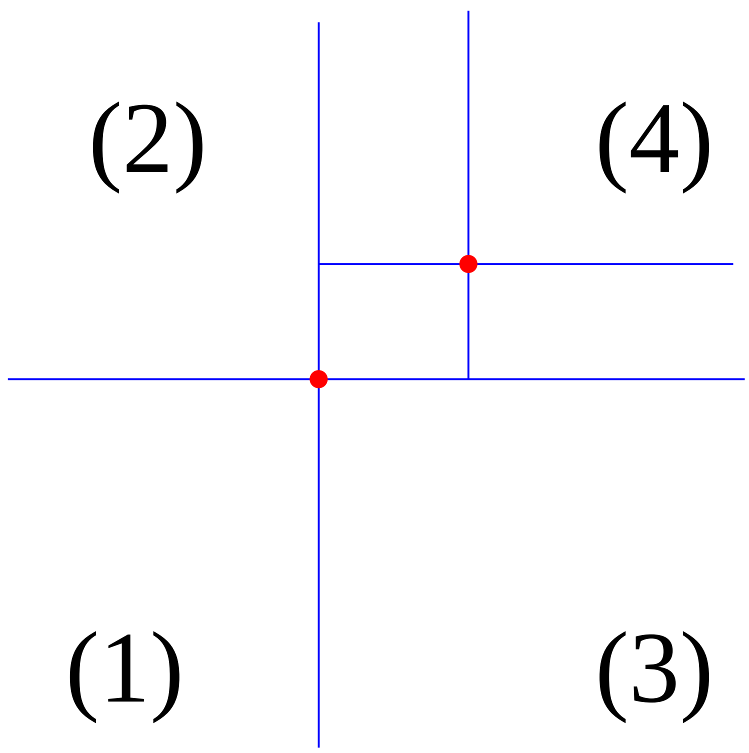

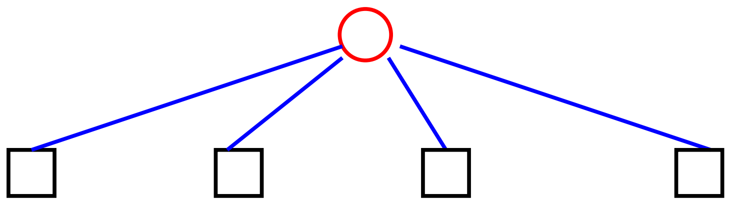

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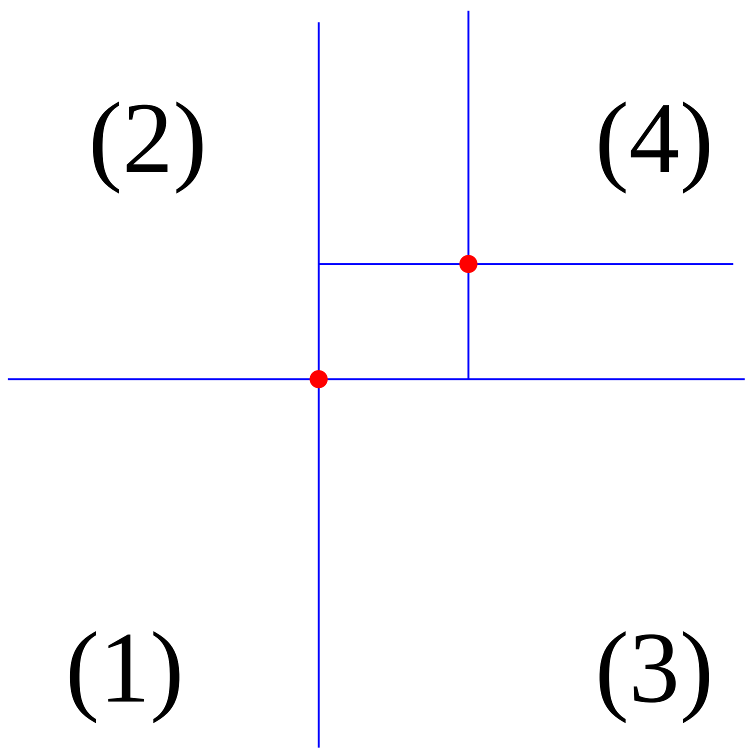

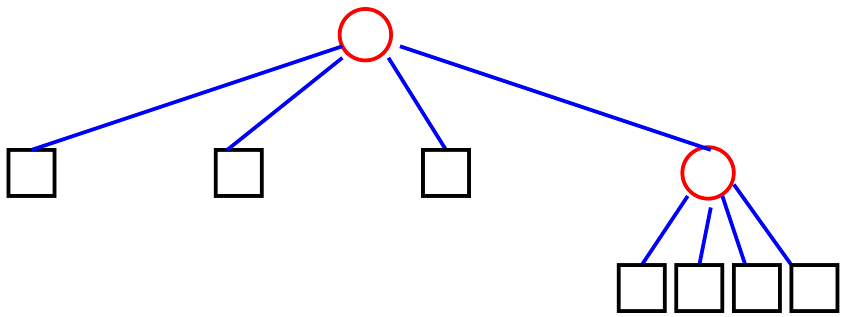

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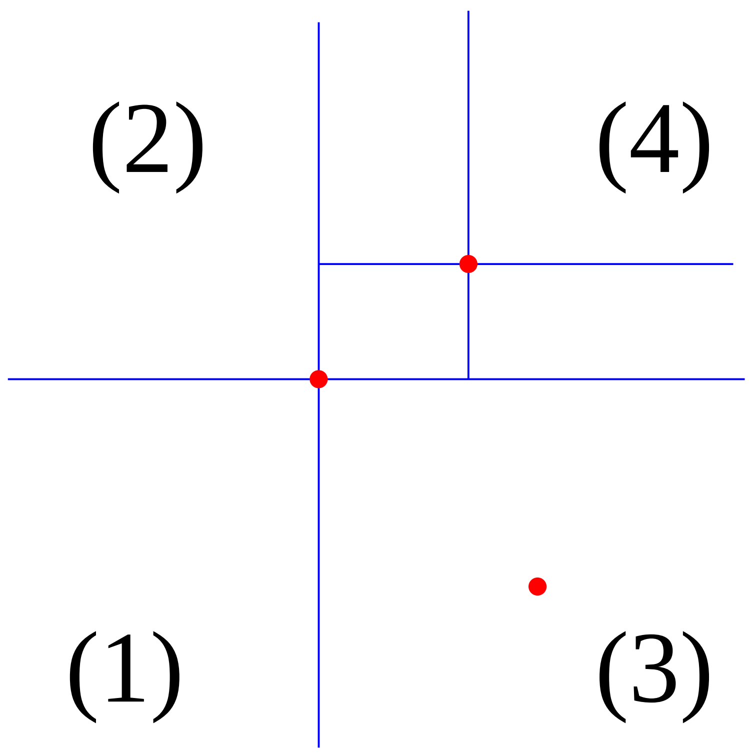

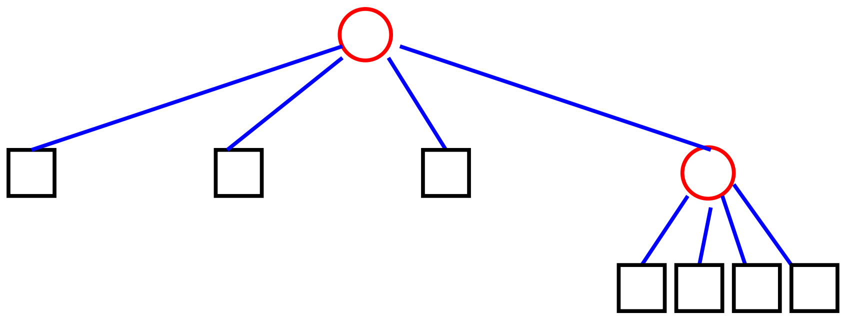

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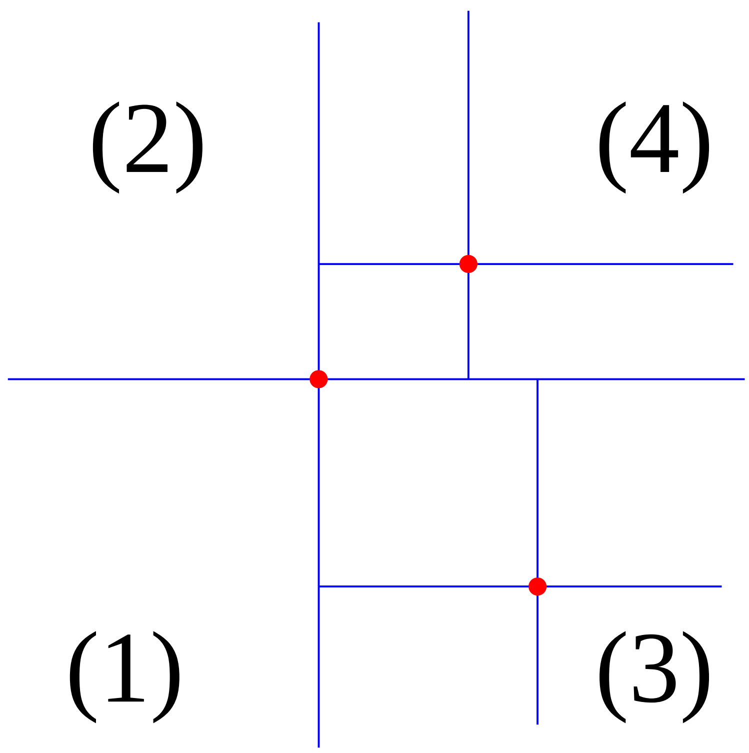

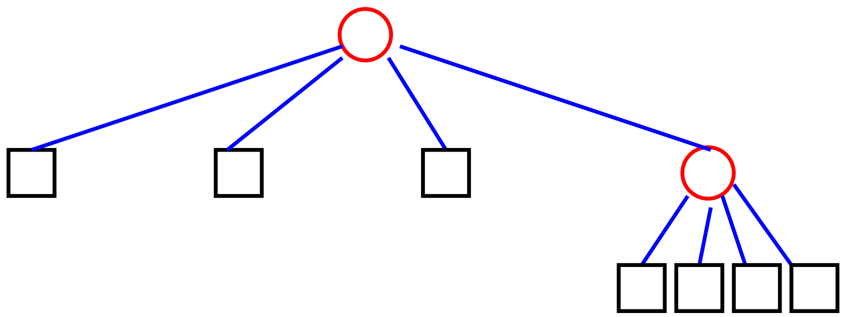

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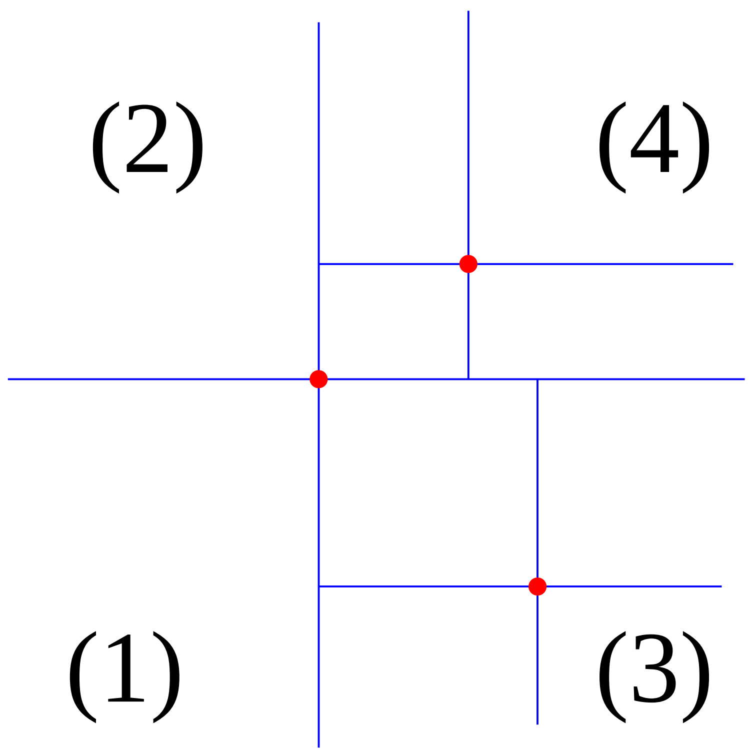

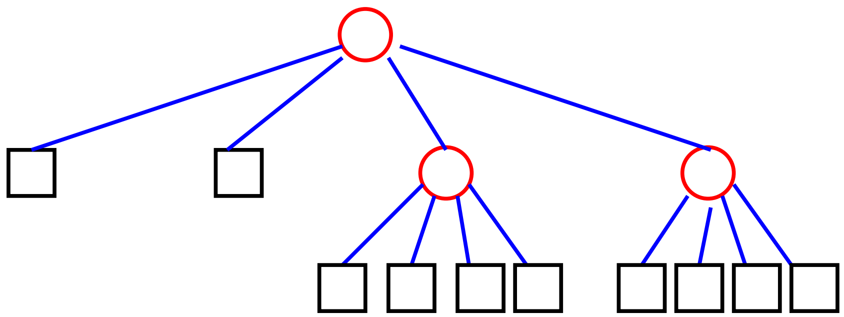

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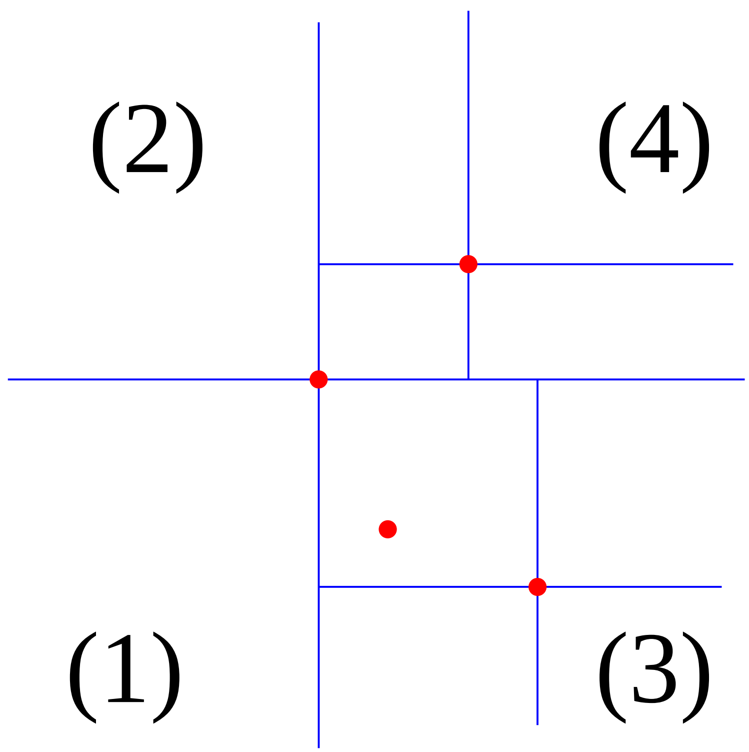

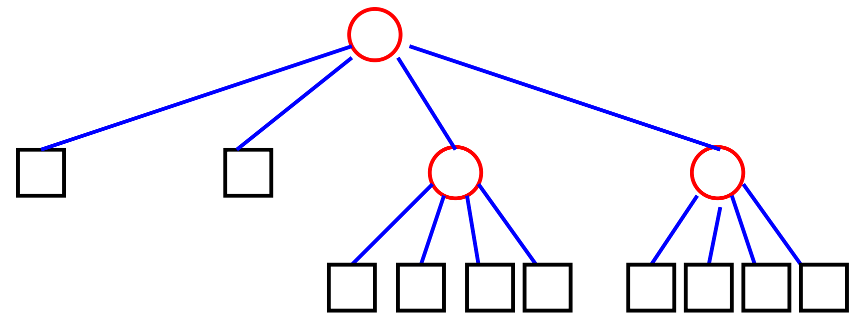

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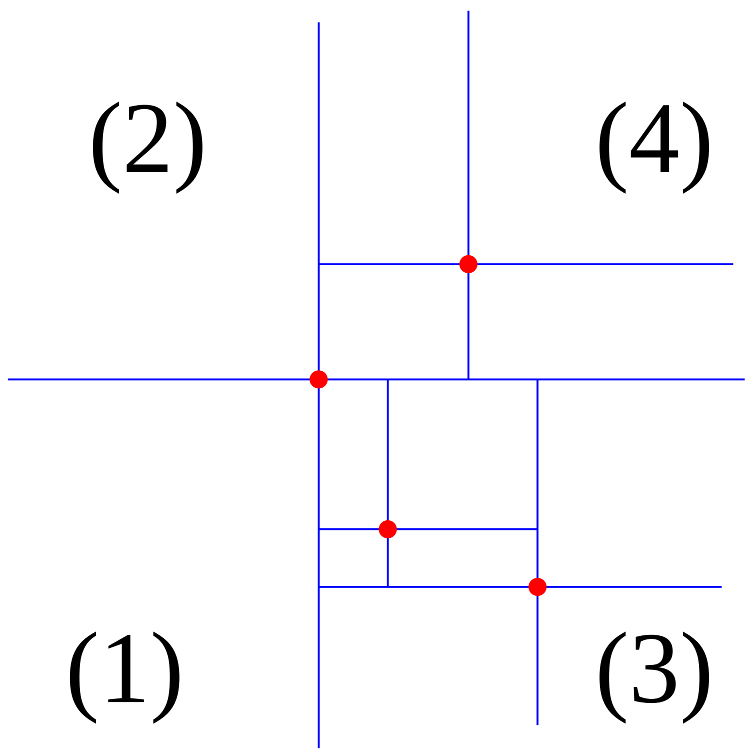

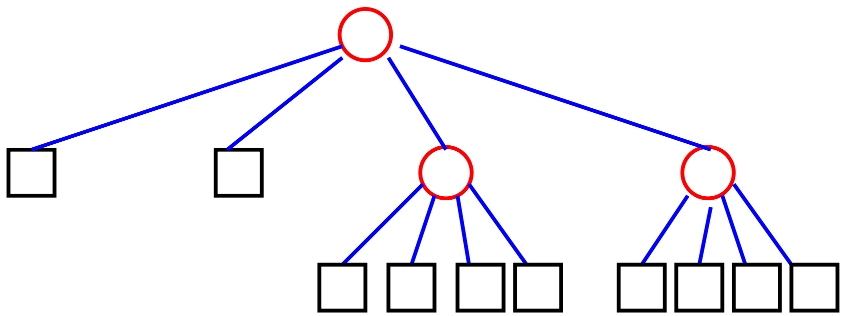

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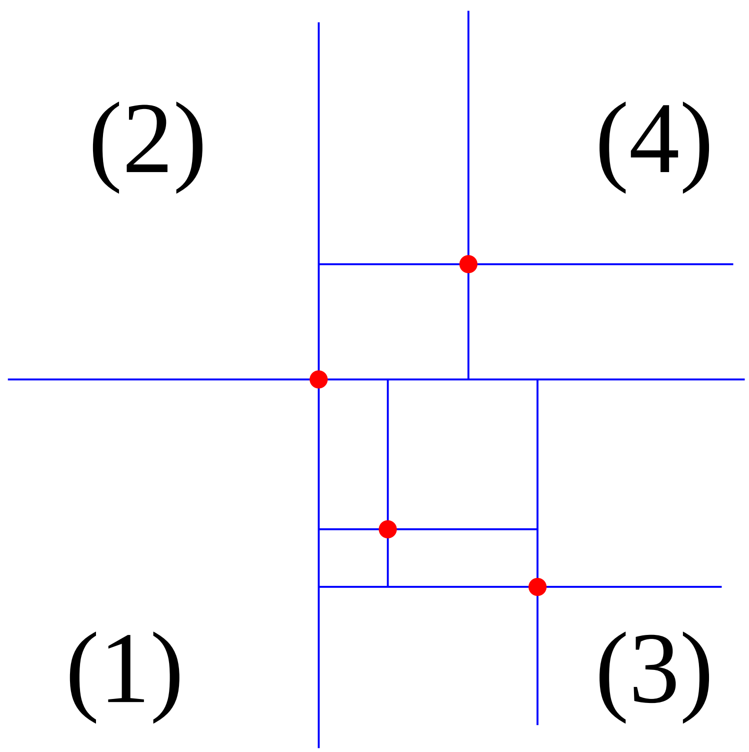

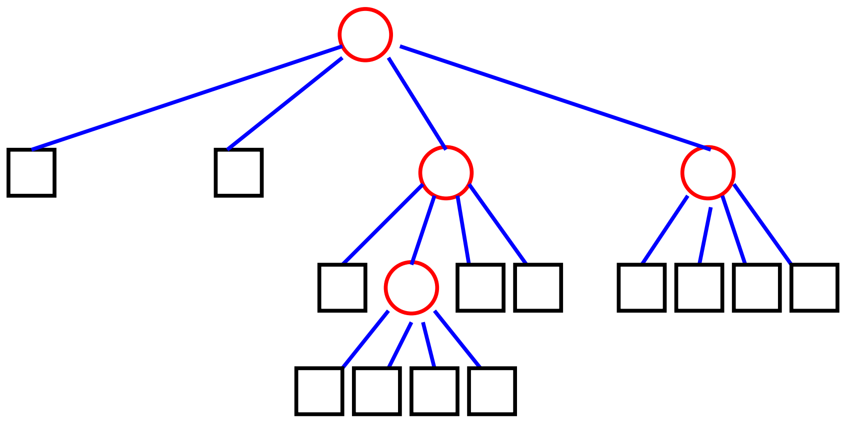

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Devroye (1987), Devroye and Laforest (1990), Flajolet, Labelle, Laforest, Salvy (1995), N. and Rüschendorf (1999), etc.

Partial match queries

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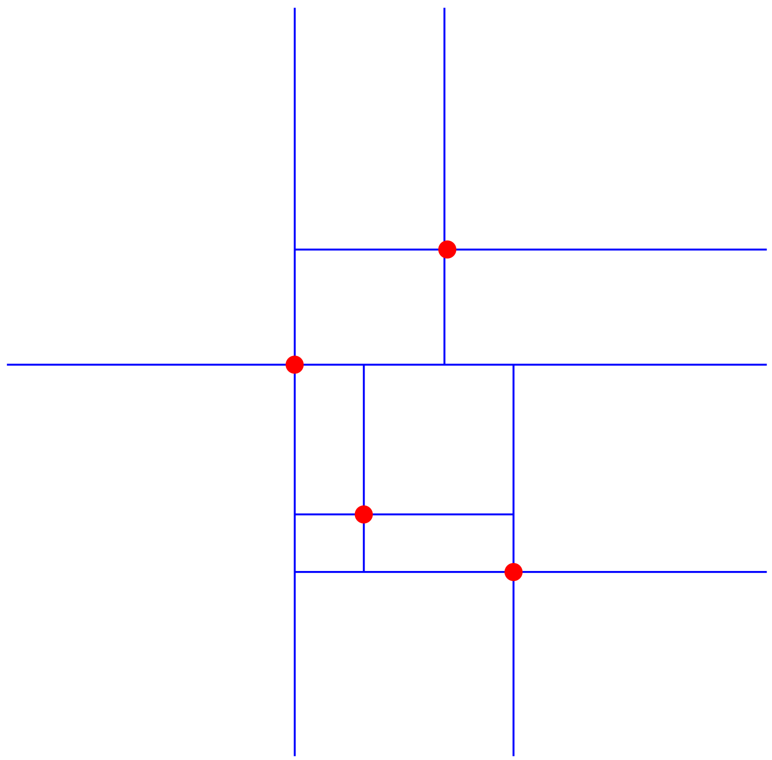

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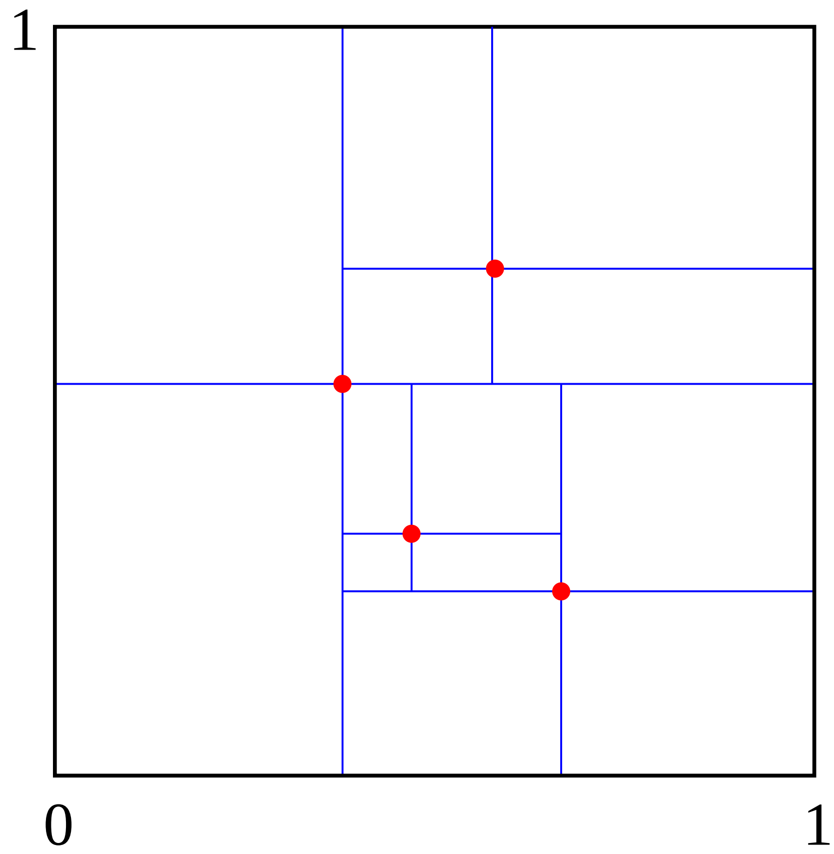

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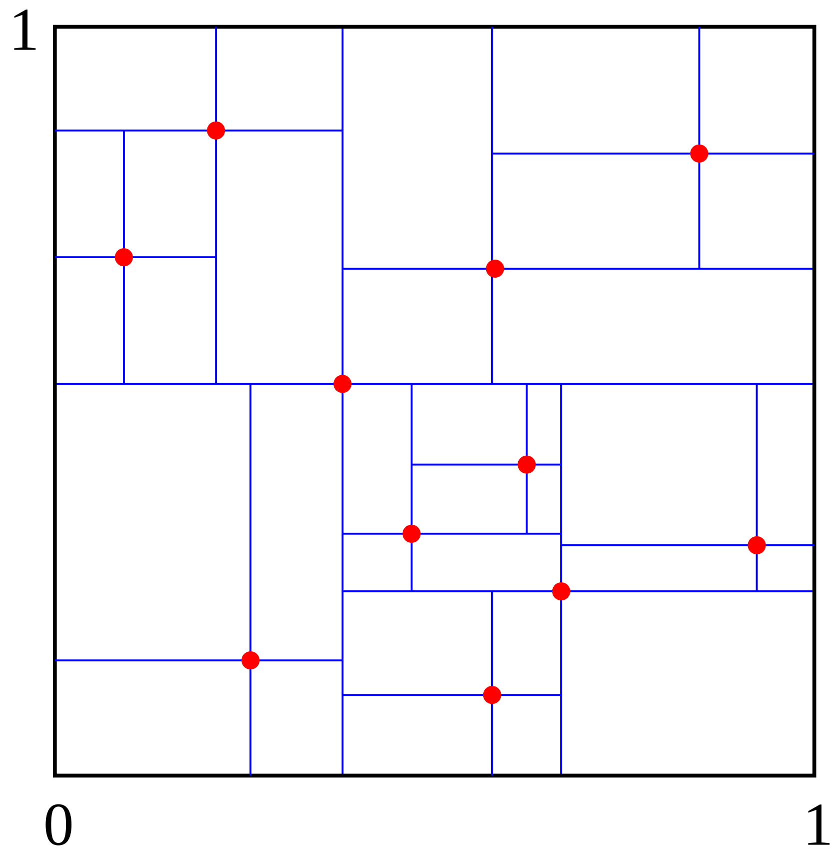

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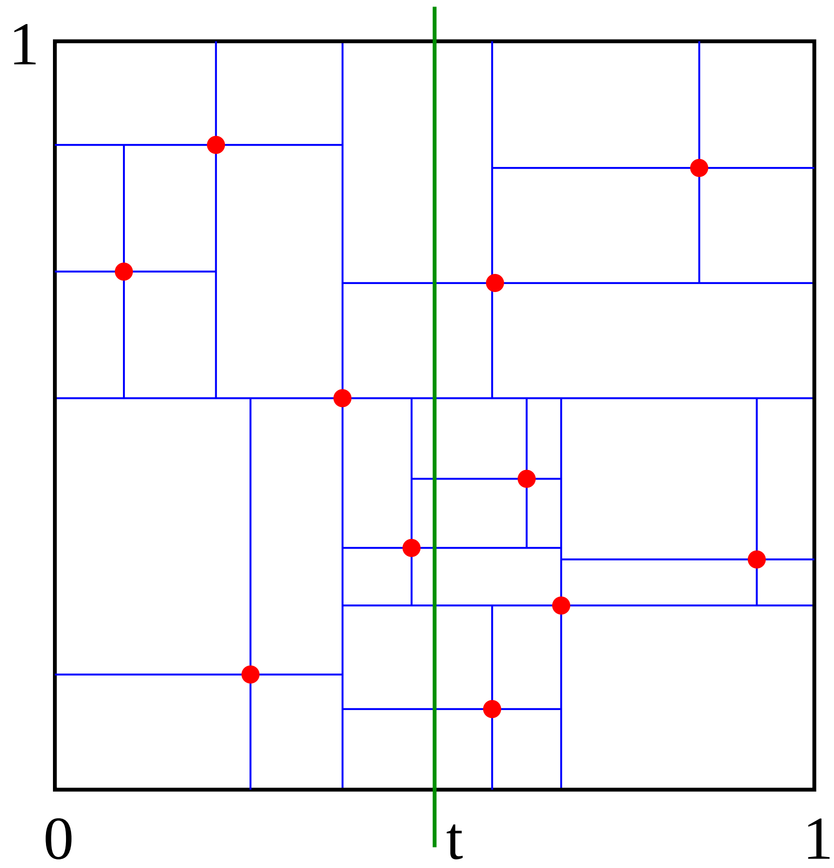

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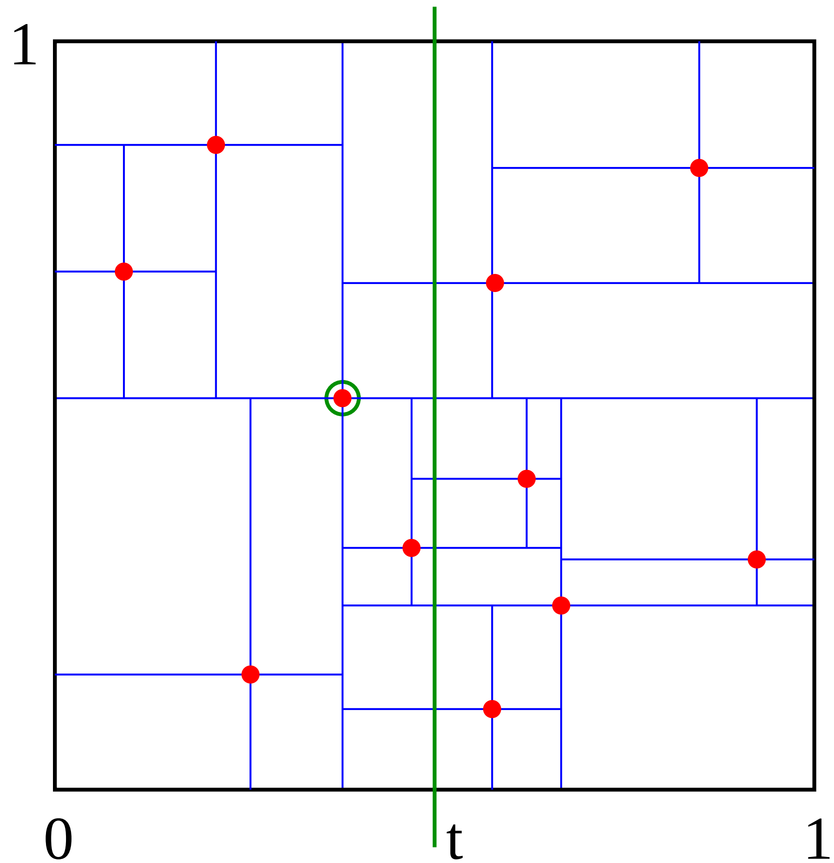

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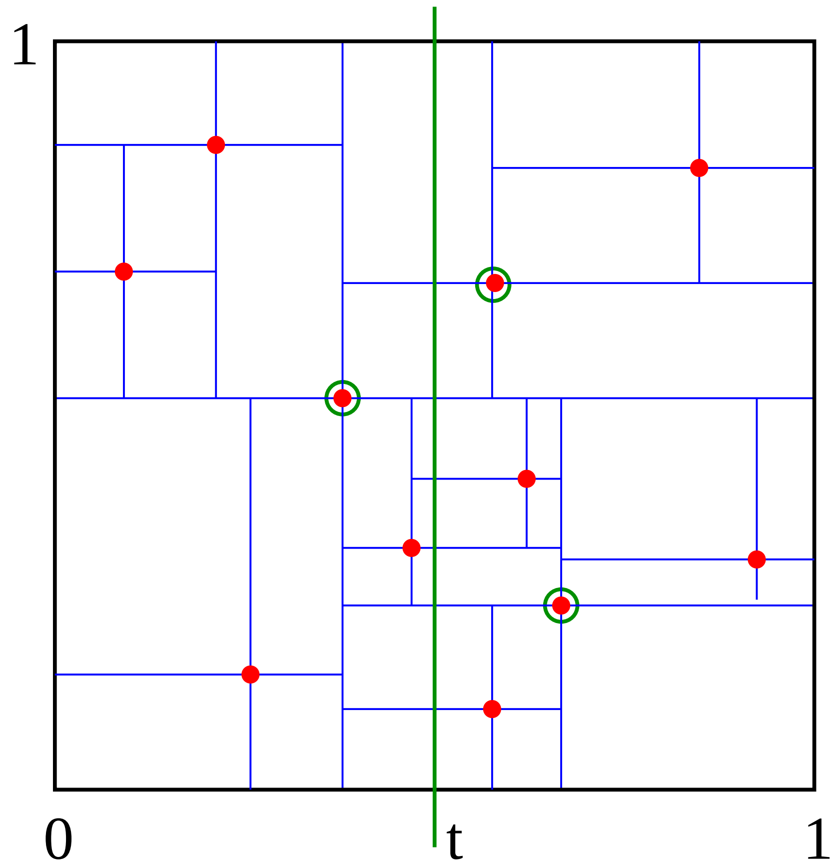

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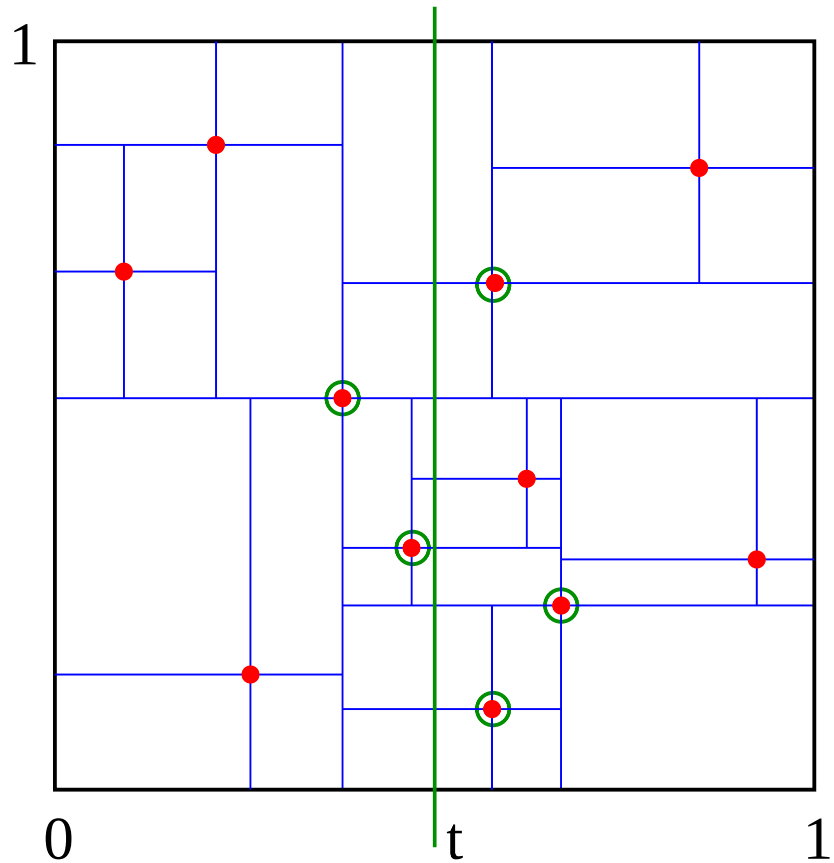

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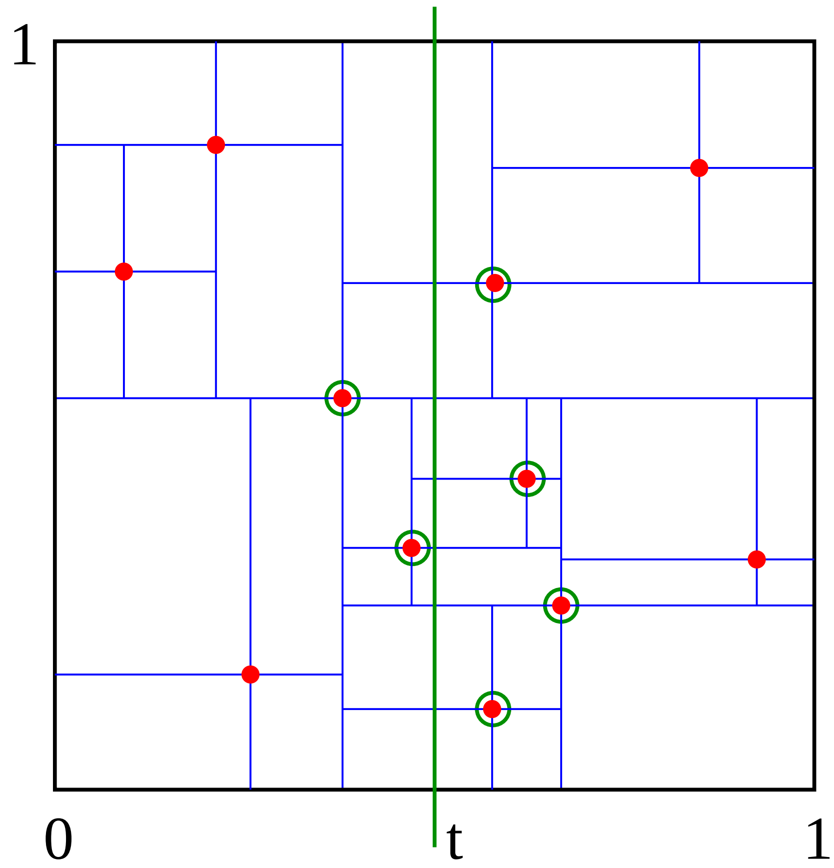

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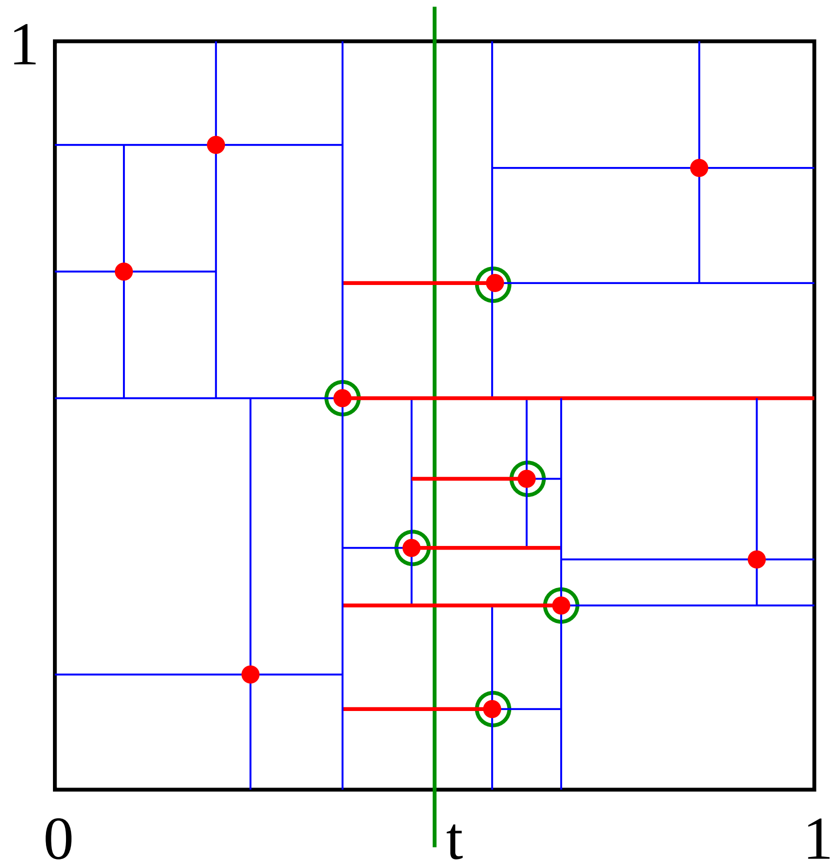

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$$C_n(t) := \#\text{ nodes visited during the query } (t, \star)$$
$$= \#\text{ number of lines cut by } x_1 = t.$$

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Z uniform on $[0, 1]$,

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with

$$\beta = \frac{\sqrt{17} - 3}{2} \approx 0.56, \qquad \kappa = \frac{\Gamma(2\beta + 2)}{2\Gamma^3(\beta + 1)} \approx 1.59.$$

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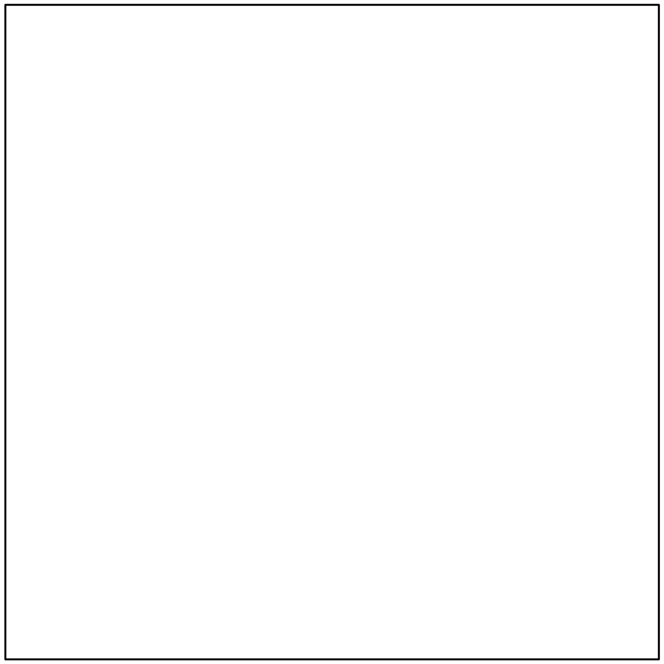
$$\frac{C_n(0)}{n^{\sqrt{2}-1}} \xrightarrow{d} W \quad \text{(nondegenerate)}$$

k-d trees



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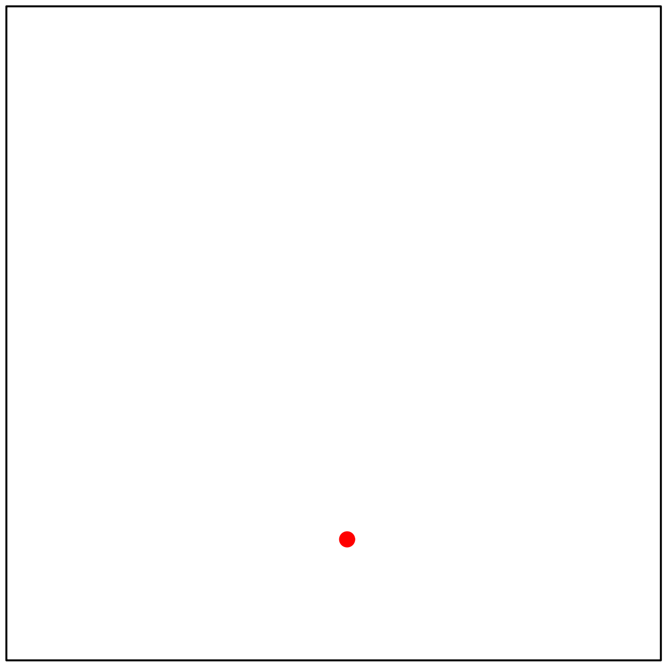

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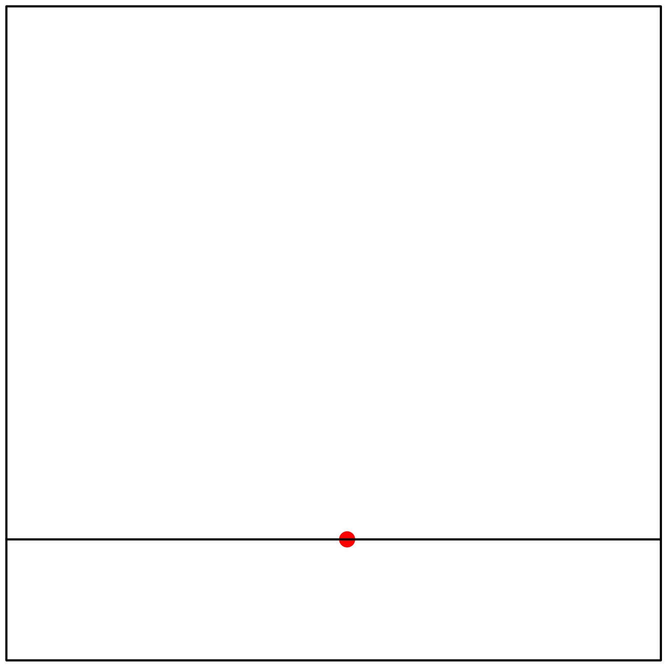

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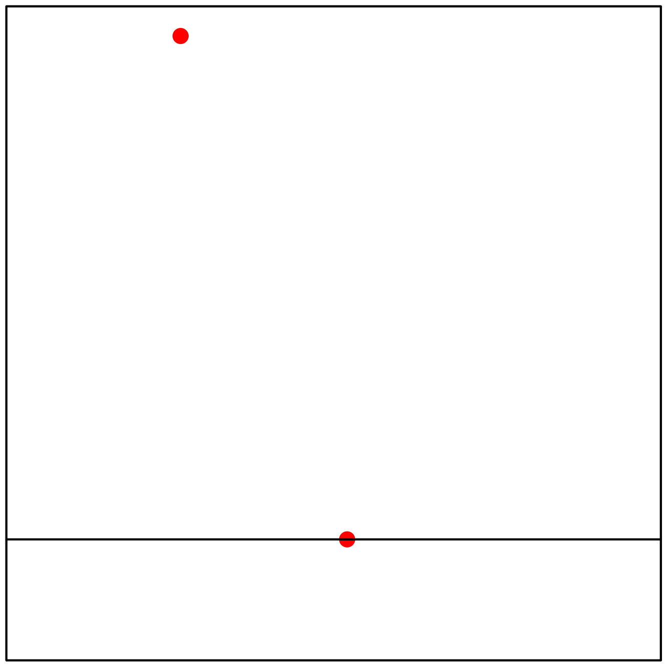

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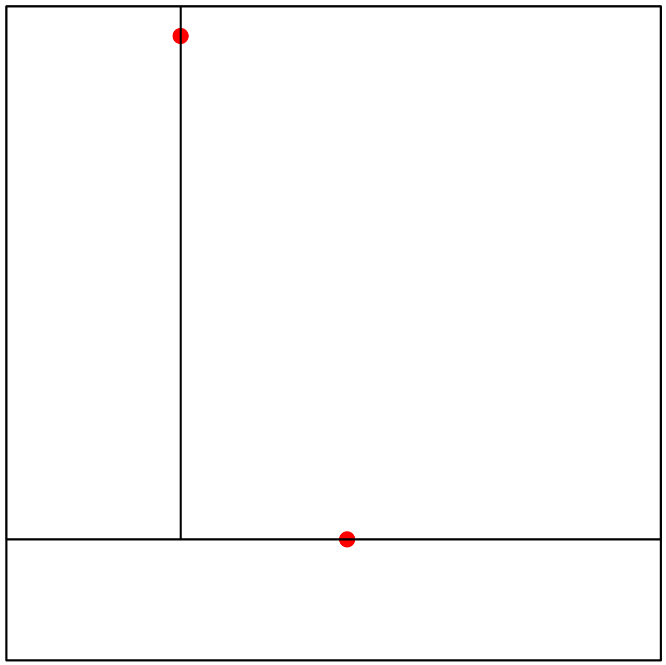

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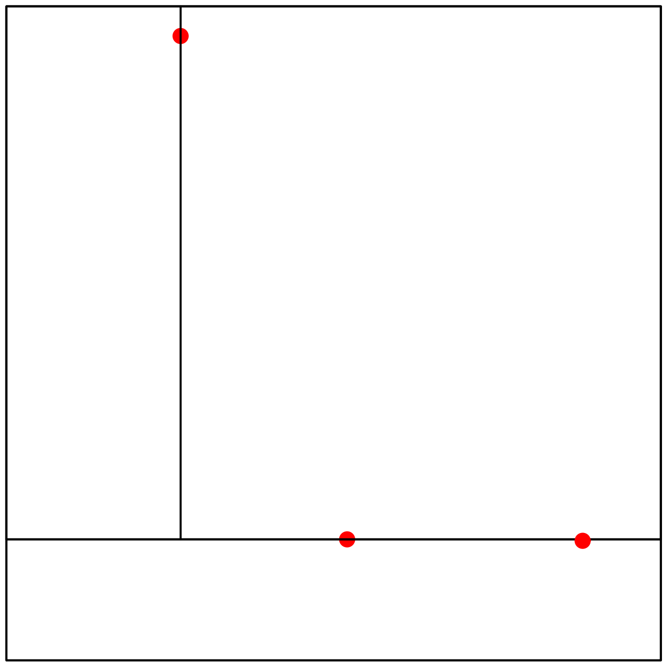

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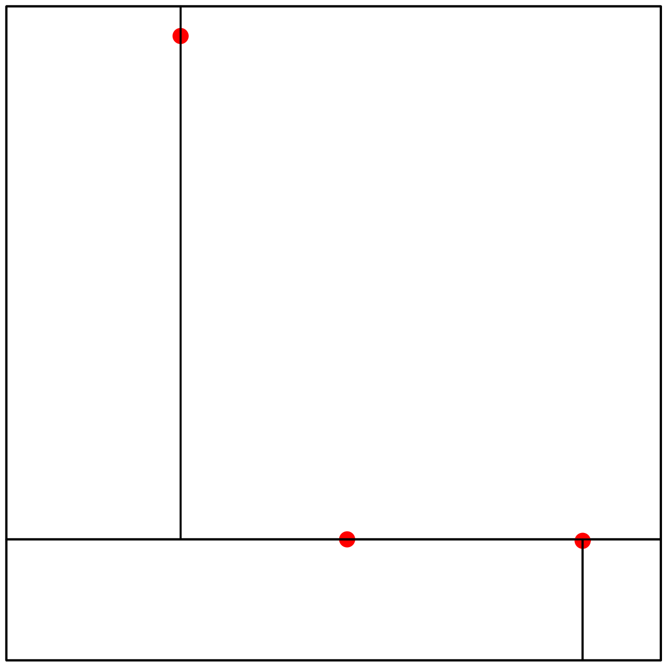

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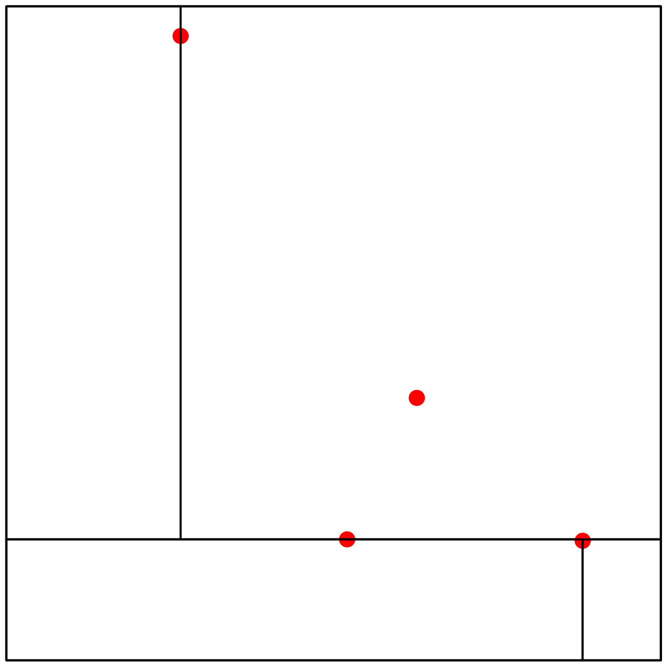

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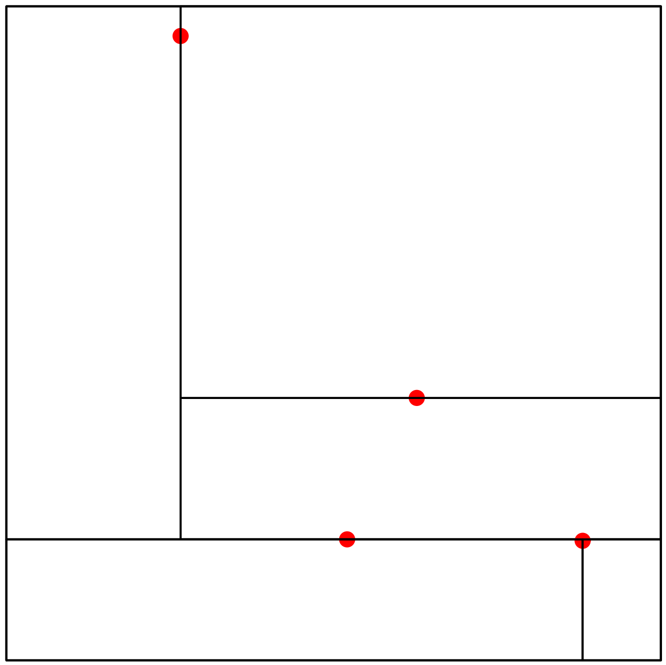

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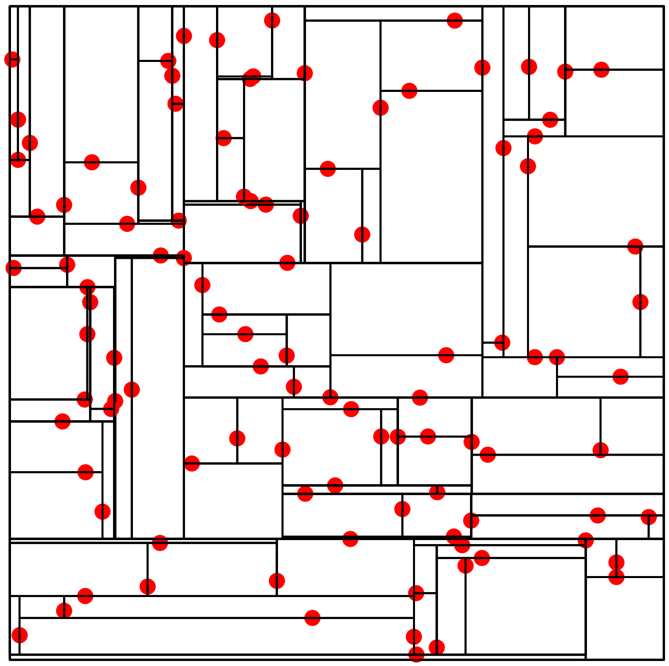

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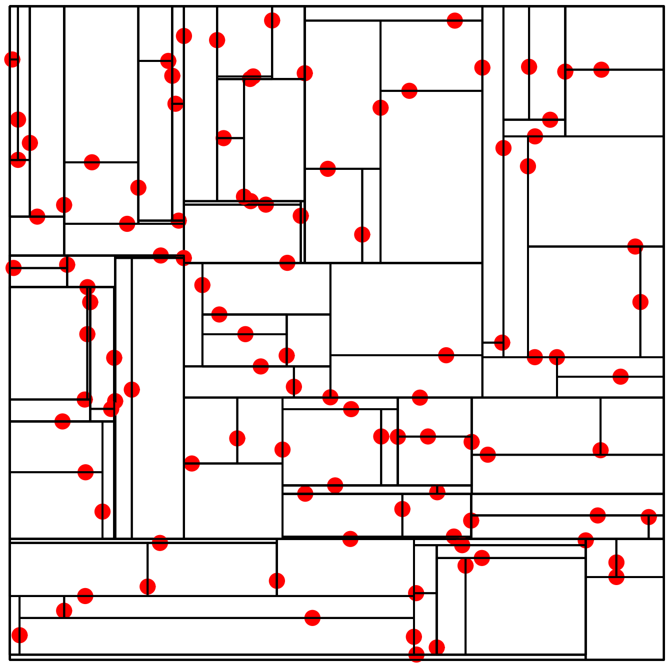

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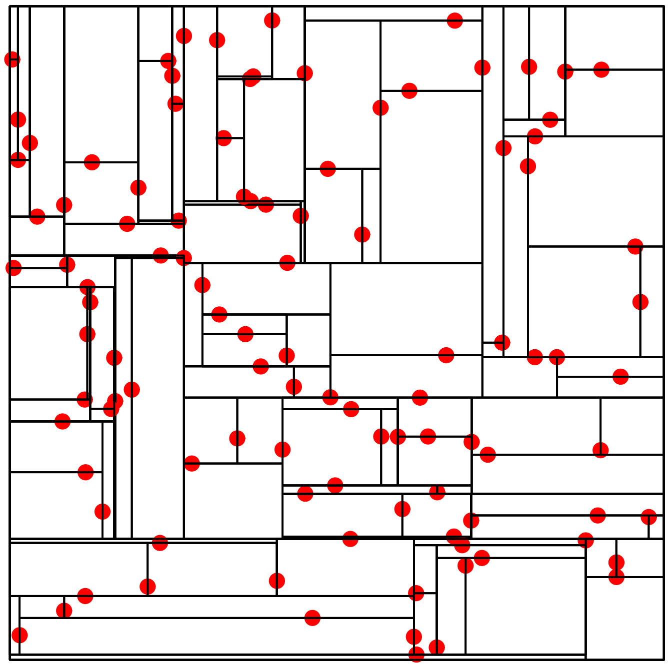

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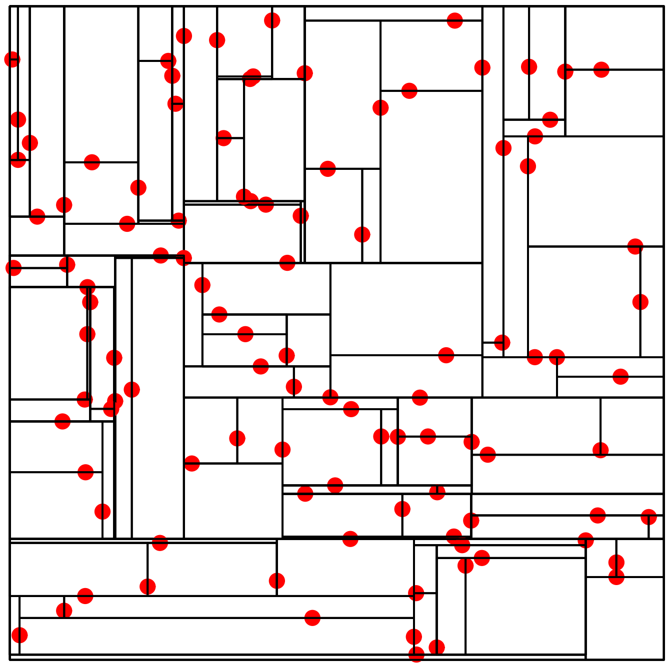

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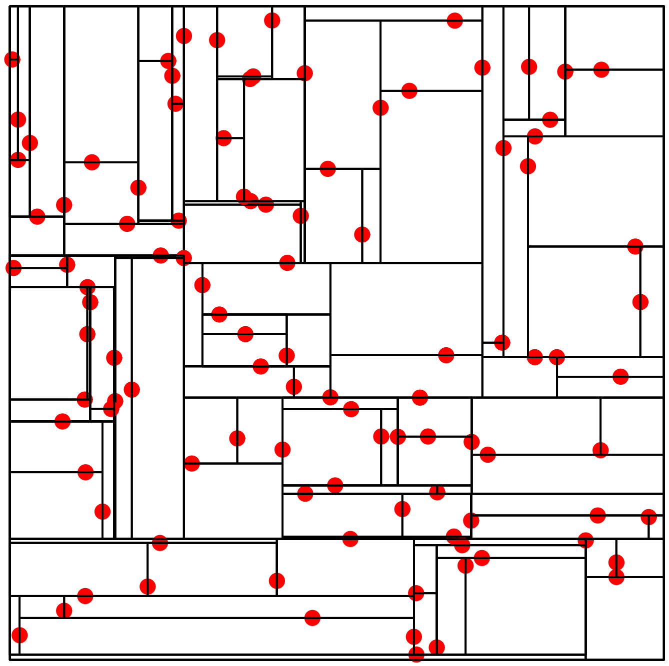

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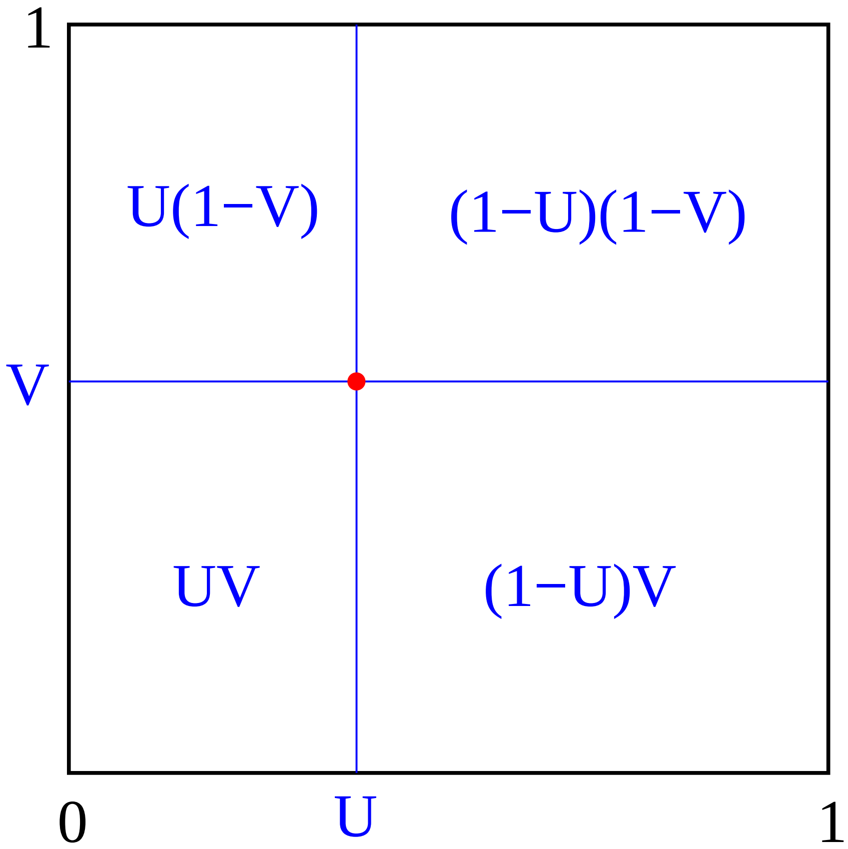
Recursive decomposition



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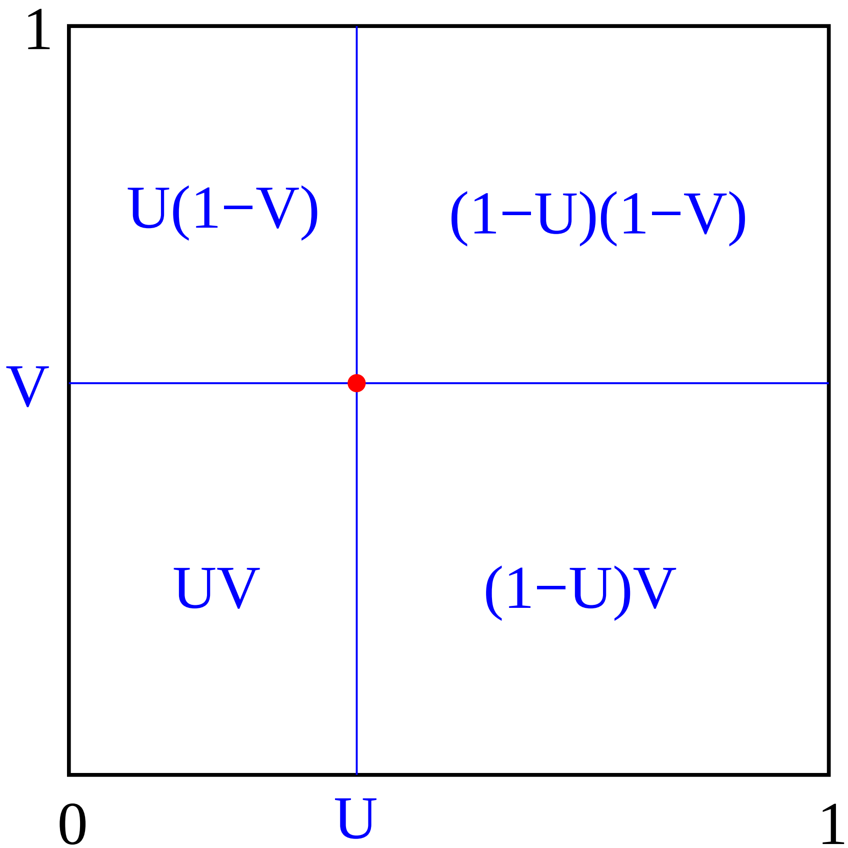

$$I^{(n)} = (I_1^{(n)}, I_2^{(n)}, I_3^{(n)}, I_4^{(n)})$$
$$\stackrel{d}{=} M(n-1; UV, U(1-V), (1-U)V, (1-U)(1-V)).$$

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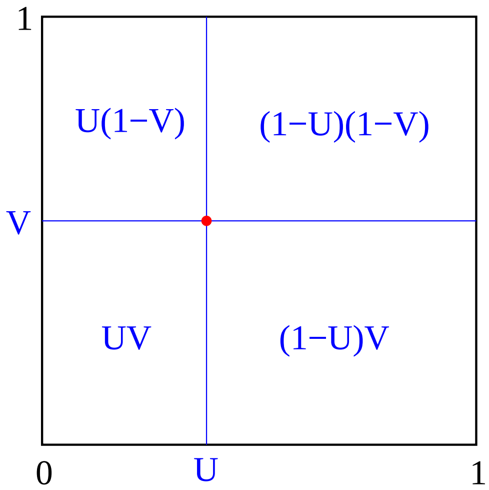

$C^{(r)}_{I^{(n)}_r}(t)$: # nodes visited in r-th sub-square (t relative to size).

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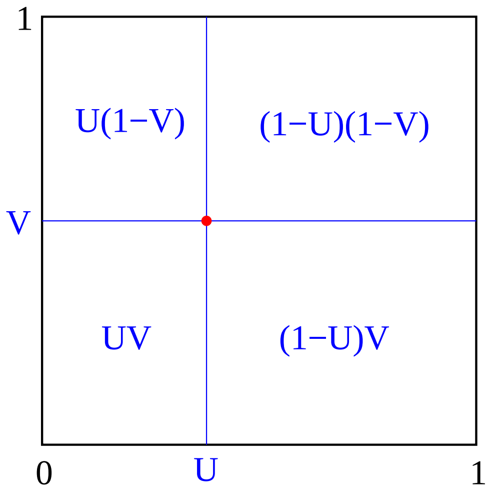

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$$\mathbf{1}_{\{s\geq U\}}\left(C^{(3)}_{I_3^{(n)}}\left(\frac{1-s}{1-U}\right) + C^{(4)}_{I_4^{(n)}}\left(\frac{1-s}{1-U}\right)\right),$$

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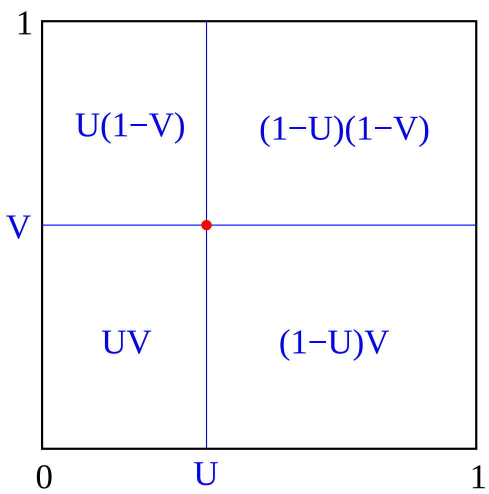

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$\left(\left(C^{(r)}_j(s)\right)_{s\in[0,1]}\right)_{j\geq 1}$, $(I^{(n)}, U)$, $r = 1, \ldots, 4$ independent

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Uniform query $x_1 = Z$

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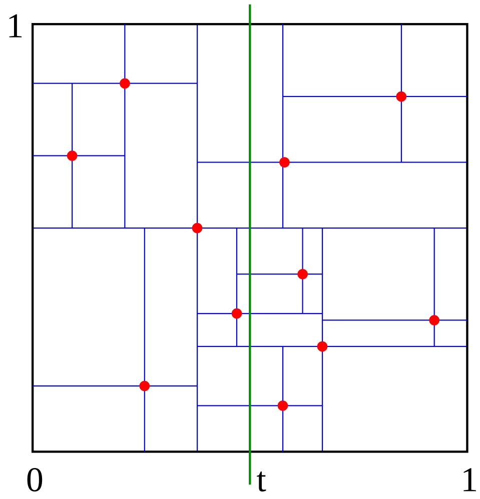
Yields a recurrence for $\mathbb{E}\, C_n(Z)$.



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Uniform query $x_1 = Z$

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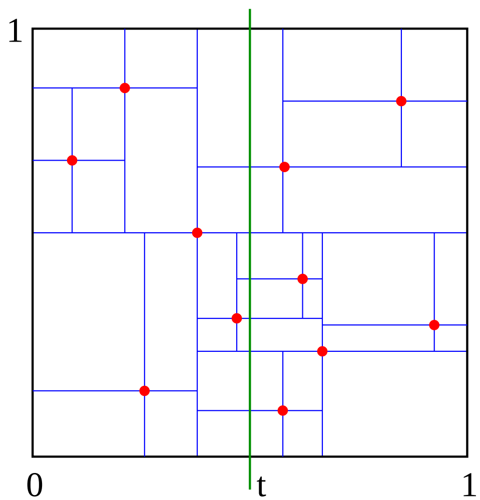
$$\mathbf{1}_{\{Z\geq U\}}\left(C^{(3)}_{I^{(n)}_3}\left(\frac{1-Z}{1-U}\right) + C^{(4)}_{I^{(n)}_4}\left(\frac{1-Z}{1-U}\right)\right).$$

Yields a recurrence for $\mathbb{E}\, C_n(Z)$.



Conditioned on $I^{(n)}$

$C^{(1)}_{I^{(n)}_1}\left(\frac{Z}{U}\right)$, $C^{(2)}_{I^{(n)}_2}\left(\frac{Z}{U}\right)$ and $C^{(3)}_{I^{(n)}_3}\left(\frac{1-Z}{1-U}\right)$, $C^{(4)}_{I^{(n)}_4}\left(\frac{1-Z}{1-U}\right)$

still dependent!!!

General recursion

$$X_n \stackrel{d}{=} \sum_{r=1}^{K} A_r(n) X^{(r)}_{I_r^{(n)}} + b_n, \qquad n > n_0.$$

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- $K \geq 1$ Number of subproblems (also $K = K_n$).
- $X_n^{(r)} \stackrel{d}{=} X_n$ (recursive).
- $I^{(n)} = (I_1^{(n)}, \dots, I_K^{(n)})$ Sizes of subproblems.
- $\left(X_n^{(1)}\right), \dots, \left(X_n^{(K)}\right), (A_1(n), \dots, A_K(n), b_n, I^{(n)})$ independent.

Contraction method

Rösler (1991, 1992)
Rachev and Rüschendorf (1995)

$$X_n \stackrel{d}{=} \sum_{r=1}^{K} A_r(n) X^{(r)}_{I_r^{(n)}} + b_n, \qquad n > n_0.$$

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$$Y_n := \frac{X_n - \mu(n)}{\sigma(n)}.$$

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with

$$\begin{aligned} A_r^{(n)} &= \frac{\sigma(I_r^{(n)})}{\sigma(n)} A_r(n), \\ b^{(n)} &= \frac{1}{\sigma(n)} \Big(b_n - \mu(n) + \sum_{r=1}^{K} A_r(n) \mu(I_r^{(n)}) \Big). \end{aligned}$$

Convergence

Idea:

$$Y_n \stackrel{d}{=} \sum_{r=1}^{K} A_r^{(n)} Y_{I_r^{(n)}}^{(r)} + b^{(n)}$$

$$\downarrow \qquad\qquad \downarrow \quad \downarrow \qquad\quad \downarrow$$

$$Y \stackrel{d}{=} \sum_{r=1}^{K} A_r^{*} \; Y^{(r)} + b^{*}$$

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\begin{array}{ccccccc}
Y_n & \overset{d}{=} & \sum_{r=1}^{K} & A_r^{(n)} & Y_{I_r^{(n)}}^{(r)} & + & b^{(n)} \\
\downarrow & & & \downarrow & \downarrow & & \downarrow \\
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\left.\begin{array}{l}
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Limit map:

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\begin{aligned}
T : \mathcal{M} &\to \mathcal{M} \\
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\end{aligned}
$$

with $(A_1^*, \ldots, A_K^*, b^*)$, $Z^{(1)}, \ldots, Z^{(K)}$ independent, $Z^{(r)} \stackrel{d}{=} \nu$.

Two metrics

Distances for $\mathcal{L}(X), \mathcal{L}(Y)$:

— minimal ℓ_p metric —

$$\ell_p(\mathcal{L}(X), \mathcal{L}(Y)) = \inf\{\|X' - Y'\|_p : X' \stackrel{d}{=} X, Y' \stackrel{d}{=} Y\}.$$

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— Zolotarev metric ζ_s —

$$\begin{aligned} \zeta_s(\mathcal{L}(X), \mathcal{L}(Y)) &= \sup_{f \in \mathcal{F}_s} |\mathbb{E}\,[f(X) - f(Y)]|, \\ \mathcal{F}_s &= \{f \in C^m(\mathbb{R}, \mathbb{R}) : \\ &\qquad \left|f^{(m)}(x) - f^{(m)}(y)\right| \leq |x-y|^\alpha\}, \\ &\quad s = m + \alpha,\ m \in \mathbb{N}_0,\ \alpha \in (0,1]. \end{aligned}$$

Contraction properties

$$\begin{aligned} T : \mathcal{M} &\rightarrow \mathcal{M} \\ \nu &\mapsto \mathcal{L}\Big(\sum_{r=1}^{K} A_r^* Z^{(r)} + b^*\Big) \end{aligned}$$

Metric	Condition	Space
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General theorem in $\mathcal{M}_2(0)$

Let $(Y_n)_{n\geq 0}$ be L^2-integrable, with $\mathbb{E}\,Y_n = 0$ for all $n \geq 0$ and (as before)

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where $\mathcal{L}(Y)$ is the fixed-point of T in $\mathcal{M}_2(0)$.

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$(C_n(s))_{s\in[0,1]}$ r.v. in $(D[0,1], \|\cdot\|_\infty)$

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Grübel (2009)

Donker's functional limit law

$V_1, V_2, \ldots$ i.i.d., $\mathbb{E}\, V_1 = 0$, $\mathrm{Var}(V_1) = 1$, $\mathbb{E}\, |V_1|^{2+\varepsilon} < \infty$.

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$$S_t^n := \frac{1}{\sqrt{n}} \left\{ \sum_{k=1}^{\lfloor nt \rfloor} V_k + (nt - \lfloor nt \rfloor) V_{\lfloor nt \rfloor + 1} \right\}, \quad t \in [0, 1],$$

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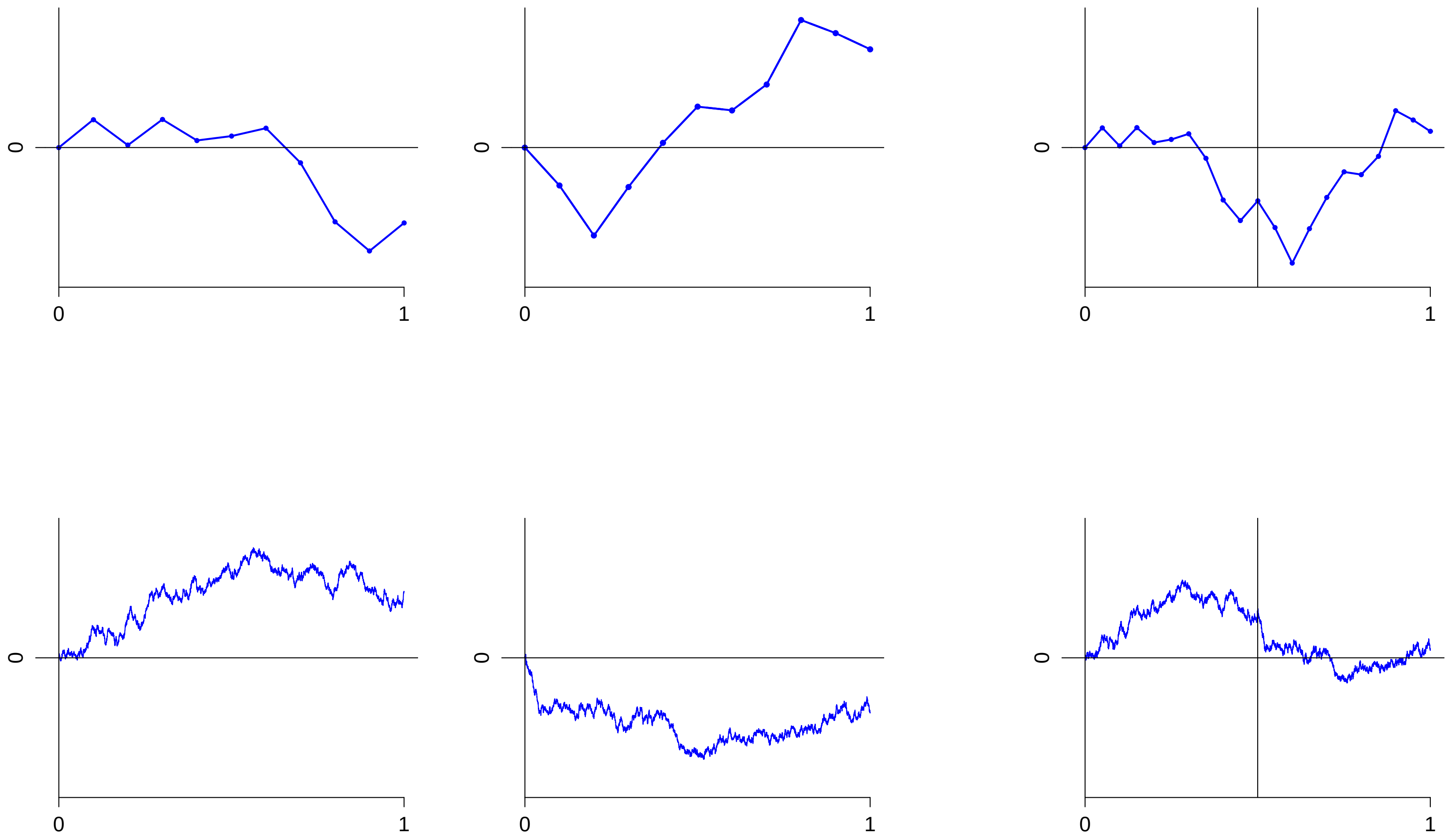
$S^n := (S_t^n)_{t \in [0,1]}$ B-valued r.v., $B = (C[0,1], \|\cdot\|_\infty)$.

Decomposition in time

Decomposition of the random walk



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For $\beta > 0$ (typically $\beta = 2$):

$$\varphi_\beta : C[0,1] \to C[0,1], \quad \varphi_\beta(f)(t) = \mathbf{1}_{\{t \le 1/\beta\}} f(\beta t) + \mathbf{1}_{\{t > 1/\beta\}} f(1),$$

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Decomposition of the Brownian motion $B = (B_t)_{t \in [0,1]}$:

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$$\begin{aligned} \zeta_s(aX, aY) &= |a|^s \zeta_s(X, Y), \quad a \in \mathbb{R}, \\ \zeta_s(X + Z, Y + Z) &\leq \zeta_s(X, Y), \quad Z \text{ indep. of } (X, Y). \end{aligned}$$

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In particular: $(X_1,Y_1),(X_2,Y_2)$ independent, then

$$\zeta_s(X_1+X_2,Y_1+Y_2) \le \zeta_s(X_1,Y_1) + \zeta_s(X_2,Y_2)$$

A formal calculation

$$\zeta_s(S^n, B) = \zeta_s\left(\sqrt{\frac{\lceil n/2\rceil}{n}}\varphi_{\frac{n}{\lceil n/2\rceil}}\left(S^{\lceil n/2\rceil}\right) + \sqrt{\frac{\lfloor n/2\rfloor}{n}}\psi_{\frac{n}{\lceil n/2\rceil}}\left(\widehat{S}^{\lfloor n/2\rfloor}\right),\right.$$

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A formal calculation

(n even)

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A formal calculation

(n even)

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Need $s > 2$.

Finiteness of ζ_s

On $B = (C[0,1], \|\cdot\|_\infty)$ or $B = (D[0,1], \|\cdot\|_\infty)$:

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Recall

$$I^{(n)} \stackrel{d}{=} M(n-1; UV, U(1-V), (1-U)V, (1-U)(1-V))$$

Fixed point equation

$$X(s) \stackrel{d}{=} \mathbf{1}_{\{s<U\}}\left((UV)^{\beta} X^{(1)}\left(\frac{s}{U}\right) + (U(1-V))^{\beta} X^{(2)}\left(\frac{s}{U}\right)\right)$$

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X has a.s. continuous paths.

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Theorem: With X the fixed point above as $n \to \infty$:

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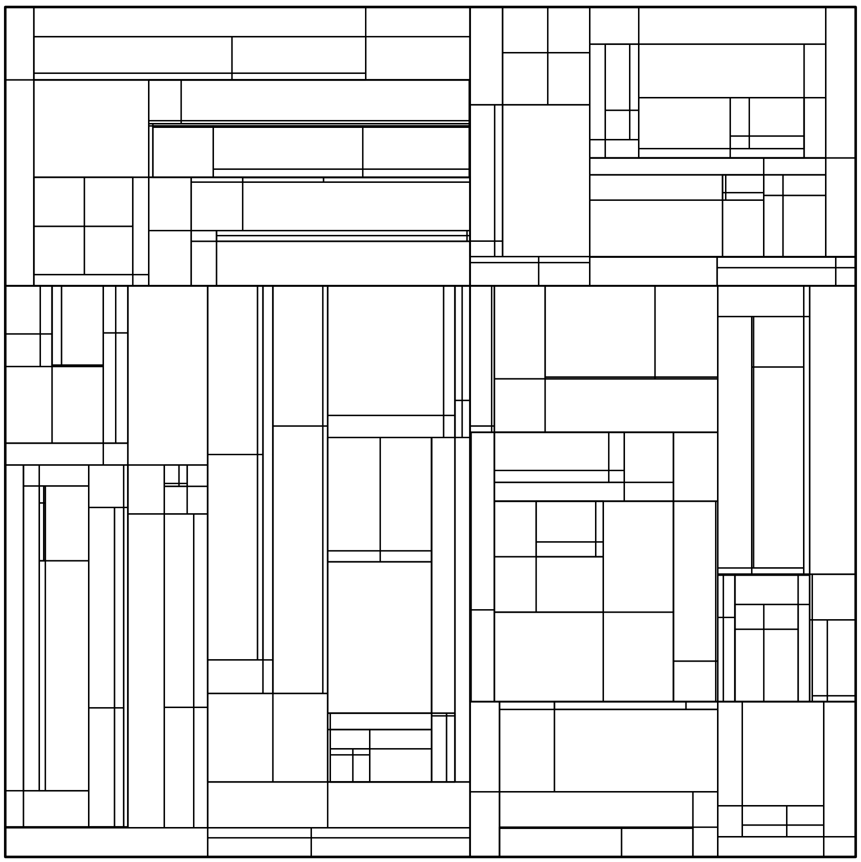

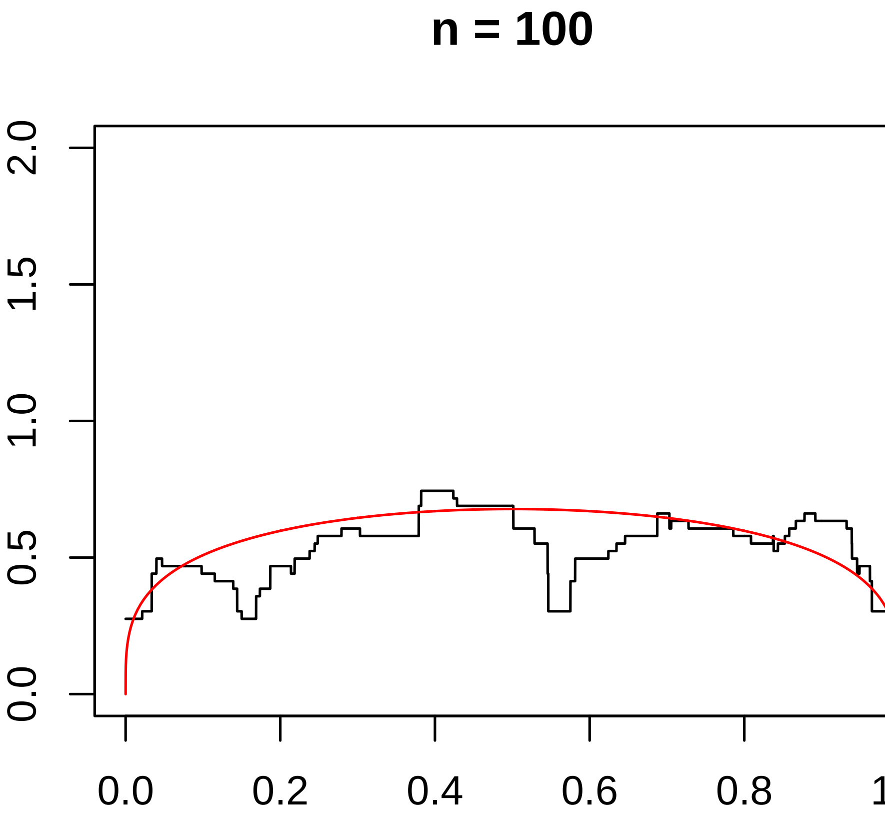

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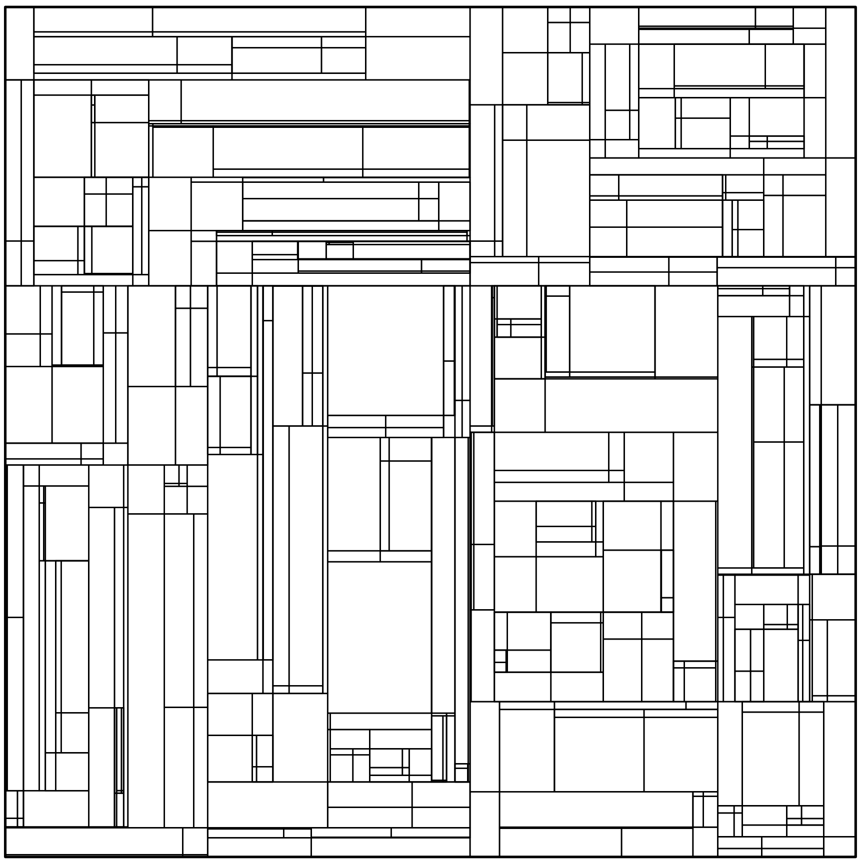

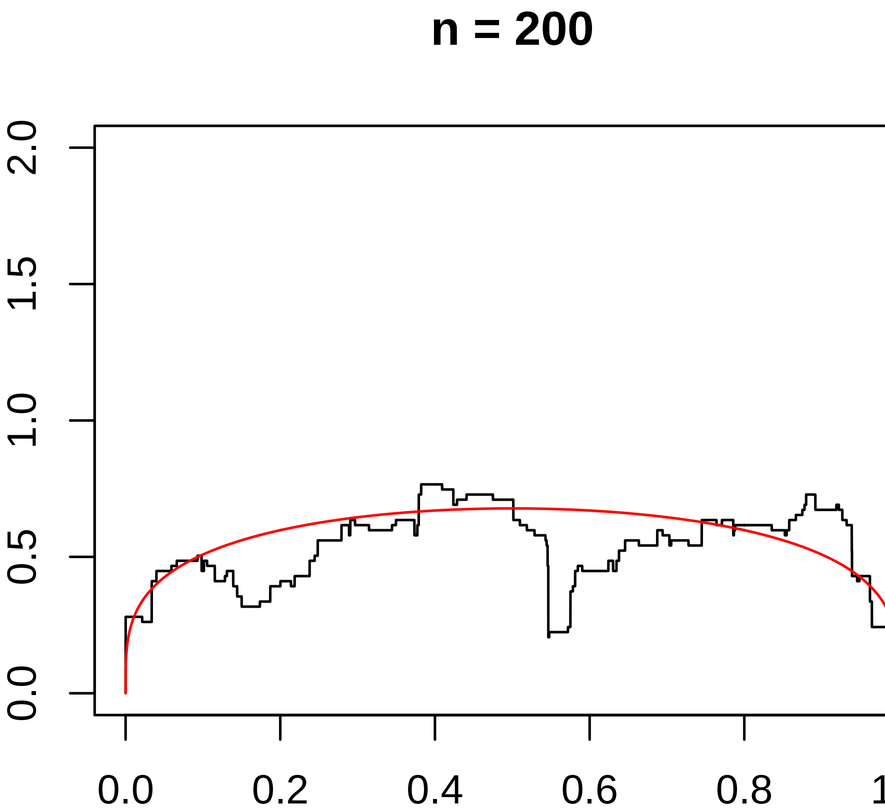

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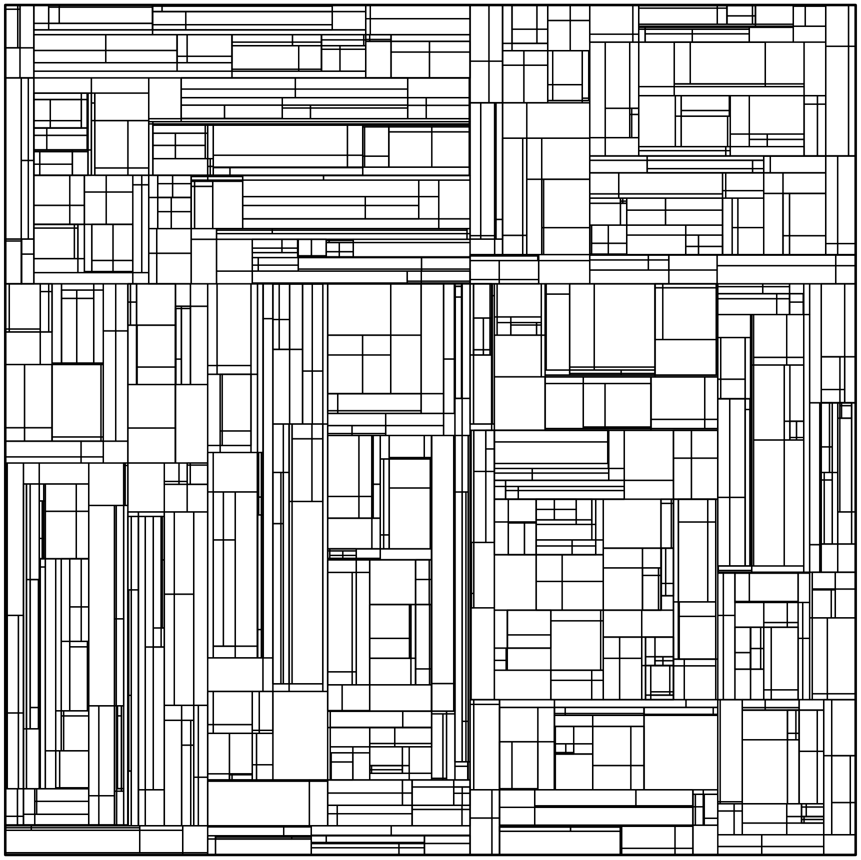

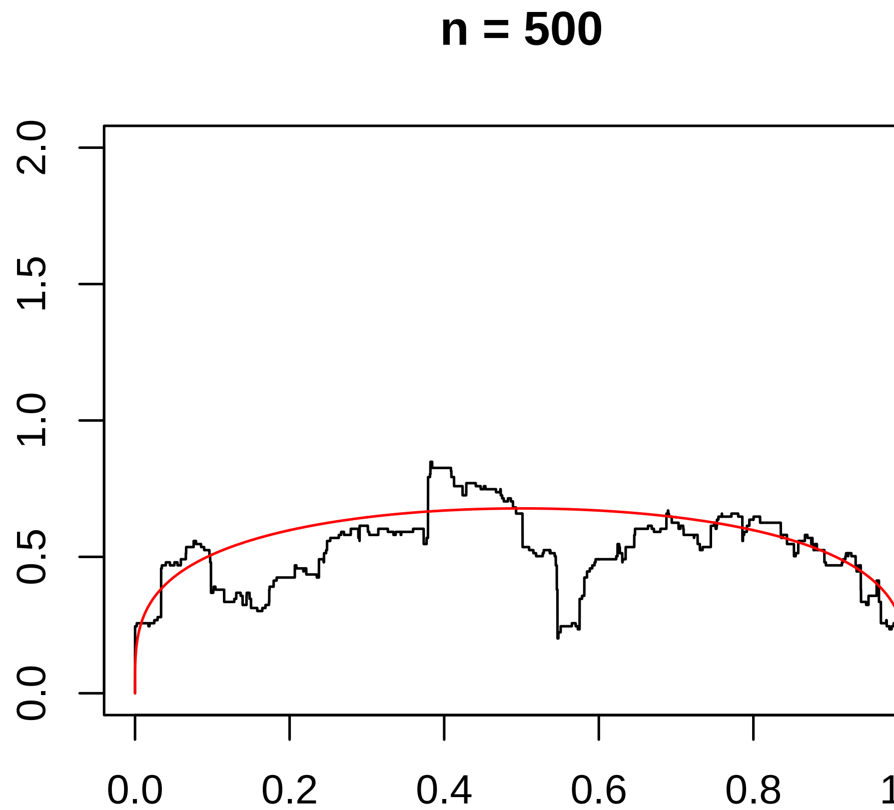

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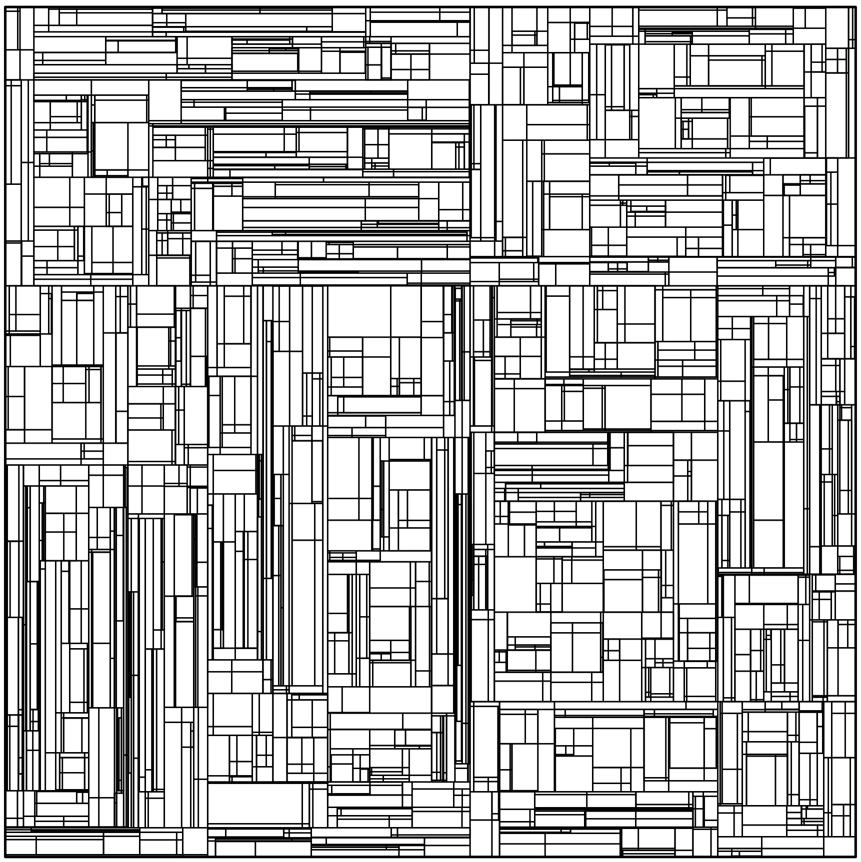

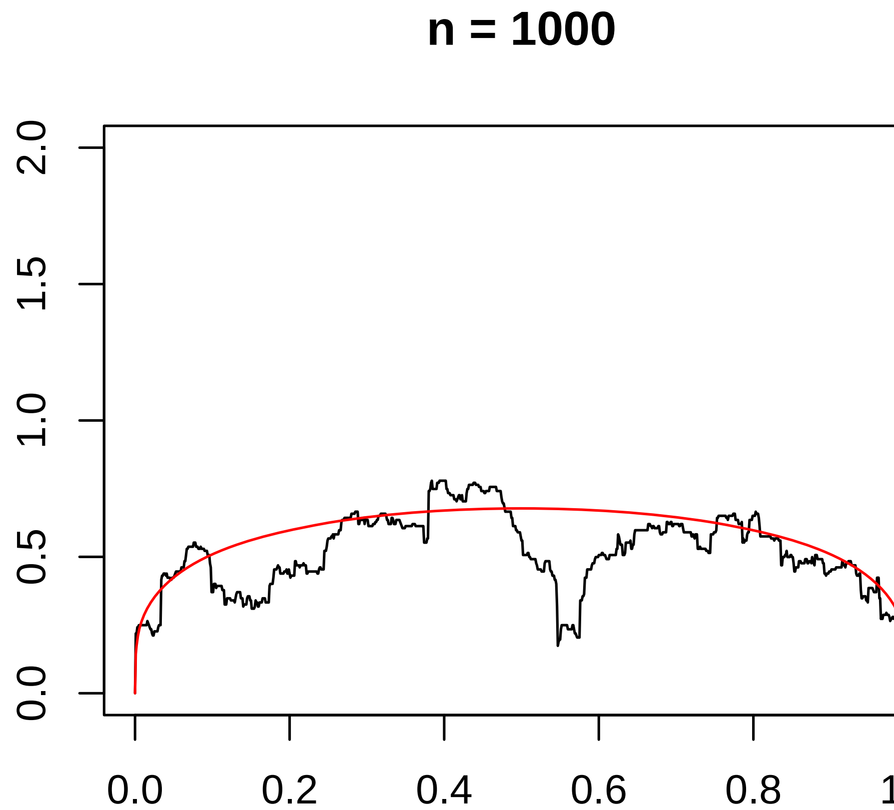

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$$K_2 = \frac{2(2\beta+1)}{3(1-\beta)} B(\beta+1, \beta+1) - 1.$$

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Query $x_1 = Z$.

Theorem: Z uniform on $[0, 1]$, independent of C_n and X. Then

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CMT: M_1, M_2 metric spaces, $h : M_1 \to M_2$ continuous.

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Moreover, (by properties of ζ_s)

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