The two-sided infinite extension of the Mallows model for random permutations

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What is a random permutation of $\mathbb{N}$? or $\mathbb{Z}_-$? or $\mathbb{Z}$?

What is a *uniformly random* permutation of $\mathbb{N}$?

A *"uniformly"* random permutation of $\mathbb{N}$ can be understood as

- a random order on $\mathbb{N}$ defined by $i \lhd j \Leftrightarrow U_i < U_j$ for U_i's iid uniform$[0, 1]$ (Rubin, Aldous, folklore $\cdots$)
- Markov chain (urn model) known as the *Chinese-restaurant process* of the growth of cycles (Dubins-Pitman)
- A limit object for the Chinese-restaurant process, which is a probability on a compactification of $\cup_{\{n \geq 1\}} \mathfrak{S}_n$, the space of *virtual permutations* (Vershik, Kerov, Olshanski, Okounkov).

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In any case, a *"uniformly" random permutation of* $\mathbb{N}$ (or $\mathbb{Z}$)

- is not a random bijection $\mathbb{N} \to \mathbb{N}$ (or $\mathbb{Z} \to \mathbb{Z}$).

What is then a good definition of a random permutation of infinite set?

A possible strategy: consider *conditionally* uniform distributions on $\mathfrak{S}_n$, with biased distribution of some permutation statistic $\texttt{stat}$, and *consistent* for $n = 1, 2, \dots$ See what happens as $n \to \infty$.

This approach has been used for $\texttt{stat}$ being the number of cycles (Ewens' permutations), cycle partitions (Kingman's partition structures), number of descents, set of descents, number/set of records, number/set of pikes. None of such constructions produced infinite random permutation as *bijection*.

In this talk the biased statistic will be the number of inversions. We shall

- recall the Mallows model for permutations of $\{1, \ldots, n\}$
- extend it to permutations (bijections) $\mathbb{N} \to \mathbb{N}$ and $\mathbb{Z}_- \to \mathbb{Z}_-$
- extend it to permutations (bijections) $\mathbb{Z} \to \mathbb{Z}$ (two-sided extension)
- discuss some distributional properties of the extensions

Randomised bubble sorting

Fix $p \in (1/2, 1)$. A permutation of n integers is created by iterations of the following procedure. Starting with any permutation,

- each pair of adjacent entries is chosen with the same probability $1/(n-1)$,
- the entries are arranged in the increasing order with probability p, and arranged in the decreasing order with probability $1-p$.

In one step,

$$21354 \text{ may change to } \mathbf{12}354, 2\mathbf{31}54, 21\mathbf{53}4, 213\mathbf{45}$$

The mixing time is $O(n^2)$ (Diaconis/Rum conjectured, Benjamini et al proved).

For $p = 1/2$ the time is $O(n^3 \log n)$ (Wilson).

The Mallows measure

The bubble sorting converges to the Mallows distribution on $\mathfrak{S}_n$

$$P(\Sigma_n = \sigma) = \frac{q^{\mathrm{inv}(\sigma)}}{[n]_q!}, \quad \sigma \in \mathfrak{S}_n$$

where

$$q = 1/p - 1 < 1,$$

$$\mathrm{inv}(\sigma) = \#\{(i,j) : i < j, \sigma(i) > \sigma(j)\}$$

is the number of inversions, $[n]_q = 1 + q + \cdots + q^{n-1}$ and $[n]_q! = [1]_q[2]_q \ldots [n]_q$.

For large n the number of inversions is only linear

$$\mathrm{inv}(\Sigma_n) \sim q(1-q)^{-1}n,$$

which is to be compared with the uniform case ($q = 1$) when $\mathrm{inv} \sim n^2/4$.

The q-exchangeability

For arbitrary initial list $v_1, \dots, v_n$ the bubble-sorting converges to Σ_n applied to $v_{(1)} \le \cdots \le v_{(n)}$. The output random word $w_1, \dots, w_n$ is characterised by a quasi-invariance property:

- if in $w_1, \dots, w_n$ the entries w_i, w_{i+1} are replaced by w_{i+1}, w_i, then the probability is multiplied by

$$q^{\mathrm{sgn}(w_{i+1} - w_i)}$$

In particular,

Mallows' measure is the unique q-exchangeable distribution on $\mathfrak{S}_n$.

Consistency

For finite integer interval $I \subset \mathbb{Z}$ Mallows permutation $\Sigma_I : I \to I$ is defined as Σ_n with $P(\Sigma_I = \sigma) = cq^{\mathrm{inv}(\sigma)}$.
Think of Σ_I as *random order* on I, which can be obviously restricted to subsets.
For finite intervals $I \subset J$ let $\pi_{J,I} : \mathfrak{S}_J \to \mathfrak{S}_I$ be relabeling of the substring $(\sigma(i),\ i \in I)$ using the increasing bijection $\{\sigma(i), i \in I\} \to I$ (ranking by I), e.g. $\pi_{[1,5],[2,4]}(4\mathbf{153}2) = 243$.

Theorem

Mallows distributions P_I are consistent w.r.t. the projections $\pi_{J,I}$. Therefore by Kolmogorov's theorem there exist extensions $P_{\mathbb{N}}$ and $P_{\mathbb{Z}}$ as measures on the space of total orders *on $\mathbb{N}$, resp. $\mathbb{Z}$.*

Proof: q-exchangeability of a random sequence $w_1, \dots, w_n$ is preserved by passing to $f(w_1), \dots, f(w_n)$ with nondecreasing f.

In the uniform case ($q = 1$) the 'uniformly random order' $\triangleleft$ on $\mathbb{N}$ or $\mathbb{Z}$ is of the type $(\mathbb{Q}, <)$, thus does not correspond to any bijection $\mathbb{N} \to \mathbb{N}$ resp. $\mathbb{Z} \to \mathbb{Z}$.

But for any other $q \neq 1$ the situation is different: the extended Mallows measures $P_{\mathbb{N}}$ resp. $P_{\mathbb{Z}}$ are supported by permutations (bijections), which displace infinitely many integers.

Left and right inversions under Mallows' n-permutation

We have

$$\mathrm{inv}(\Sigma_n) = \sum_{j=1}^{n} L_j = \sum_{j=1}^{n} R_j.$$

with left/right inversions counts

$$L_j := \#\{i < j : \Sigma_n(i) > \Sigma_n(j)\}, \quad R_j := \#\{i > j : \Sigma_n(i) < \Sigma_n(j)\}.$$

- L_j are independent, truncated geometric with $P(L_j = k) = q^k/[k]_q$, $0 \le k < j \le n$
- R_j are independent, truncated geometric with $P(R_j = k) = q^k/[n-j]_q$, $0 \le k \le n-j$.

The one-sided extension of Mallows model, $\Sigma_{\mathbb{N}}$

In the $n \to \infty$ limit $R_1, R_2, \dots$ are i.i.d. geometric, $L_1, L_2, \dots$ independent truncated geometric.

Constructing $\Sigma_{\mathbb{N}}$ as the *q-shuffle* of $1, 2, \dots$ (aka absorption sampling or Russian roulette: A. Kemps, Rowling):

- let $\Sigma(1) := R_1 + 1$, eliminate $\Sigma_{\mathbb{N}}(1)$ from $1, 2, \dots$ Inductively, set $\Sigma_{\mathbb{N}}(j)$ to be the $(R_j + 1)$st minimum of the remaining integers, and delete $\Sigma(j)$ from the list.

Example: $R_1 = 2, R_2 = 3, R_3 = 2 \dots$ yields permutation starting with $3, 5, 4, \dots$

Application: a q-analogue of de Finetti's theorem

Theorem

The most general infinite q-exchangeable sequence is obtainable by applying $\Sigma_{\mathbb{N}}$ to some (unique) random sequence $v_1 \leq v_2 \leq \ldots$.

Finite-dimensional distributions of $\Sigma_{\mathbb{N}}$

Represent $\sigma : \mathbb{N} \to \mathbb{N}$ as 0-1-matrix with 1's in positions $(j, \sigma(j))$. For $k \times k$ minor let I be the set of rows with a 1, and let J the set of columns with a 1.
Under $P_{\mathbb{N}}$ the probability of such minor with $\#I = \#J = r$ 1's is

$$(1-q)^r q^{k^2 - 2kr - k + \mathrm{inv}_k + \sum_{i \in I} i + \sum_{j \in J} j},$$

where inv_k is the number of inversions $(i, \sigma(i)), (j, \sigma(j))$ in the $k \times k$ minor, meaning $i < j \le k, \quad k \ge \sigma(i) > \sigma(j)$.

Balanced permutations of $\mathbb{Z}$

Permutation (bijection) $\sigma : \mathbb{Z} \to \mathbb{Z}$ is *tame* if

$$\ell_j := \#\{i < j : \sigma(i) > \sigma(j)\} < \infty,$$

$$r_j := \#\{i > j : \sigma(i) < \sigma(j)\} < \infty,$$

then it has finite *balance*

$$b := \#\{j \geq 0 : \sigma(j) < 0\} - \#\{j < 0 : \sigma(j) \geq 0\}.$$

A permutation is *balanced* if $b = 0$.

Lemma

Each tame permutation satisfies $\sigma(j) = j - \ell_j + r_j - b$.

The ranking algorithm

For tame σ the counts of right inversions are

$$r_j = \sum_{j=i}^{\infty} 1(\rho_{i,j} = \rho_{i,j+1}),$$

where the array $(\rho_{i,j})$ is computable recursively from the counts of left inversions $(\ell_j,\ j \in \mathbb{Z})$ as

$$\rho_{i,i} = \ell_i, \quad \rho_{i,j+1} = \rho_{i,j} + 1(\rho_{i,j} \geq \ell_{j+1}).$$

$\rho_{i,j} + 1$ for $i \leq j$ is the *rank* of $\sigma(i)$ among $\dots, \sigma(j-1), \sigma(j)$ (rank 1 means maximum); the rank is updated via ℓ_{j+1} and eventually grows to some terminal value.

Hence a balanced σ is determined by the ℓ_j's via $\sigma(j) = j - \ell_j + r_j$.

Construction of the two-sided extension

Applying the ranking algorithm to a iid geometric$(1-q)$ sequence $(L_j,\ j \in \mathbb{Z})$ yields a balanced *Mallows permutation of* $\mathbb{Z}$, denoted further Σ^*, with distribution P.

Symmetries:

- $(\Sigma^*(j) - j, \quad j \in \mathbb{Z})$ is a stationary process.
- $(\Sigma^*)^{-1} \stackrel{d}{=} \Sigma^*$.

Characterisation:

Theorem

Every q-exchangeable distribution on tame permutations of $\mathbb{Z}$ *is of the form* $B + \Sigma^*$*, where the balance parameter* $B \in \mathbb{Z}$ *is a random variable independent of* Σ^**.*

Bubble sorting in continuous time

A permutation of $\mathbb{Z}$ evolves by the rules: at unit probability rate every pair of adjacent entries is chosen, then the entries are

- arranged in increasing order with probability p,
- arranged in decreasing order with probability $1 - p$.

If the initial permutation is balanced, the long-run limit is Σ^*, the two-sided infinite Mallows permutation.

Connection to ASEP

As in Wilson and Benjamini et al: the substitution

$$\sigma(j) \to 1(\sigma(j) \geq 0)$$

maps the continuous-time bubble sorting in asymmetric exclusion process (0=hole, 1=particle). Then Σ^* is mapped to a *blocking* measure (as in Liggett).

Encode 0-1-configuration into Young diagram Λ, by counting the right inversions at 1's :

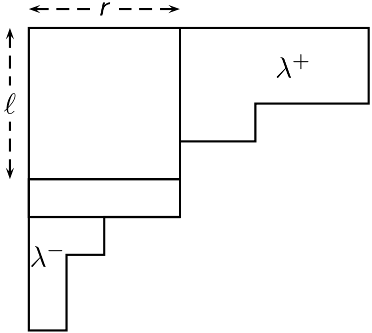
$$\dots 0000101011011111 \dots \mapsto (3, 2, 1, 1)$$

Using Eulerian generating function for partitions and q-exchangeability:

$$P(\Lambda = \lambda) = \langle \infty \rangle q^{|\lambda|},$$

where $|\lambda| = \sum_k \lambda_k$ and

$$\langle m \rangle := \prod_{k=1}^{m} (1 - q^m), \qquad \text{for} \quad m \leq \infty.$$



Decomposition of the diagram yields for the two-sided Mallows measure

$$P(R_j = r | L_j = \ell) = \langle\infty\rangle q^{\ell r + r} \left(\sum_{\lambda_+} q^{|\lambda+|} \right) \left(\sum_{\lambda_-} q^{|\lambda_-|} \right)$$

Distribution of displacement

Theorem

For $d,\ j \in \mathbb{Z}$

$$P(\Sigma^*(j) - j = d) = (1-q)\langle\infty\rangle \sum_{\{\ell, r \geq 0 : r - \ell = d\}} \frac{q^{r\ell + r + \ell}}{\langle r\rangle\langle\ell\rangle}.$$

The distribution is symmetric about 0 and has exponentially decaying tails. Moreover

$$\limsup_{n\in\mathbb{Z}} \frac{|\Sigma^*(n) - n|}{\log_{1/q}|n|} \leq 1. \quad \text{a.s.}$$

Joint distribution of displacements

For $k = 1, 2, \dots$ and $d_1 \leq d_2 \cdots \leq d_k$

$$P(\Sigma^*(1) - 1 = d_1, \dots, \Sigma^*(k) - k = d_k) =$$
$$\langle \infty \rangle \langle d_2 - d_1 \rangle \dots \langle d_k - d_{k-1} \rangle (1-q)^k q^{-k(k+1)/2}$$
$$\sum \frac{q^{\sum_{1 \leq i \leq j \leq k} (b_i+1)(a_j+1)}}{\langle b_1 \rangle \dots \langle b_k \rangle \langle a_1 \rangle \dots \langle a_k \rangle},$$

where the summation is over all nonnegative integer variables $b_1, a_1, \dots, b_k, a_k$ which satisfy the constraints

$$(a_m + \cdots + a_k) - (b_1 + \cdots + b_m) = d_m \quad \text{for} \quad m = 1, \dots, k.$$

For general $d_1, \dots, d_k$ multiply by the factor $q^{\mathrm{inv}(d_1+1, \dots, d_k+k)}$.

Interlacing construction of Σ^* from one-sided extensions

- create a unique q-exchangeable balanced sequence W of 0's and 1's
- shuffle $1, 2, \ldots$ by one-sided $\Sigma_{\mathbb{N}}$ and use them to replace all 1's in W,
- shuffle $\ldots, -1, 0$ by one-sided $\Sigma_{-\mathbb{Z}}$ and use them to replace all 0's in W.

Do each step independently. For example

$$\ldots 0\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0 | 1\ 0\ 1\ 1\ 1\ 1 \ldots$$

$$\Sigma_{\mathbb{N}} = 1\ 3\ 4\ 2\ 6\ 8\ 7\ 5\ \ldots$$

$$\Sigma_{-\mathbb{Z}} = \cdots -9\ \ -6\ \ -8\ \ -2\ \ -4\ \ -3\ 0\ \ -5$$

yield

$$\Sigma^* = \cdots -9\ \ -6\ \ -8\ 1\ \ -2\ \ -4\ 3\ \ -3\ 0 | 1\ \ -5\ 4\ 2\ 6\ 8 \ldots$$

Open questions

- two sided q-de Finetti: what is the most general q-exchangeable sequence $(w_j,\ j \in \mathbb{Z})$?
- decay of the correlation function of Σ^*?
- what are other stationary measures for continuous time bubble sort?

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