VLMC
(Variable Length Markov Chains)
Riemannian Comb and Suffix Tries

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AofA, June 2011

Outline

- Variable Length Markov Chains defined by a probabilized context tree;
- A VLMC defines a probabilistic dynamical source
- NSC existence and unicity of a stationary measure for the infinite comb;
- Mixing properties for the infinite comb
- Suffix tries
- Open questions

Biblio

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Comets, Fernandez, Ferrari (AAP, 2002) : “Processes with long memory”

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Clement-Flajolet-Vallee (Algorithmica, 2001) “Dynamical sources in Information Theory: Analysis of general tries”

Alphabet $\mathcal{A} = \{0, 1\}$ $\qquad \mathcal{L} = \mathcal{A}^{-\mathbb{N}} =$ left-infinite words

$$\begin{array}{rcl} U_0 & = & \ldots \; X_{-1}X_0 \\ U_1 & = & \ldots \; X_{-1}X_0X_1 \\ \ldots & & \\ U_n & = & \ldots \; X_{-1}X_0X_1 \ldots X_n \\ U_{n+1} & = & \ldots \; X_{-1}X_0X_1 \ldots X_nX_{n+1} = U_nX_{n+1} \end{array}$$

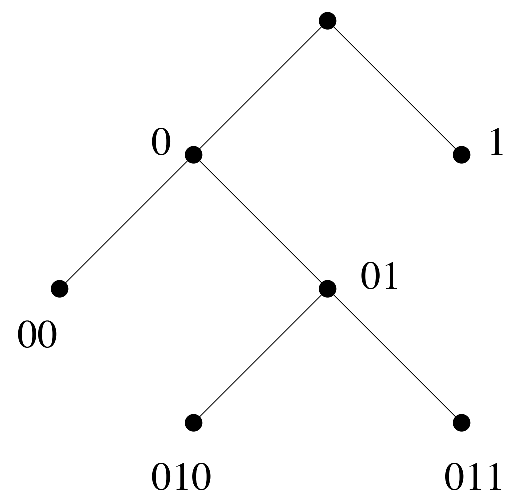
$(U_n)_{n \geq 0}$ is an $\mathcal{L}$-valued Markov chain

Start from U_0 chosen according to an initial probability measure.
From U_n to U_{n+1} add a letter X_{n+1} on the right of U_n.

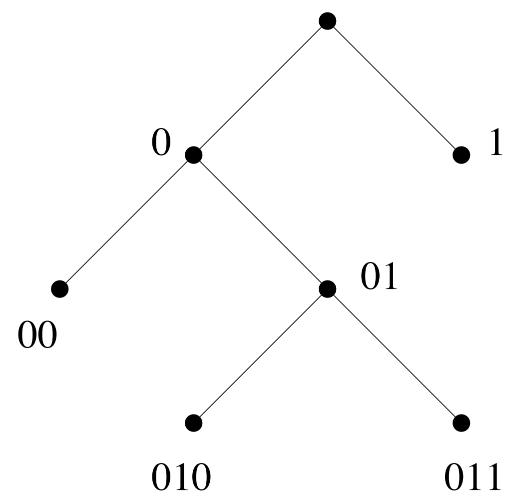
The transition from U_n to U_{n+1} is

$$\mathbb{P}(U_{n+1} = U_n\alpha | U_n)$$

where $\alpha \in \mathcal{A}$ is a letter,
it depends on a suffix of U_n. Which suffix?

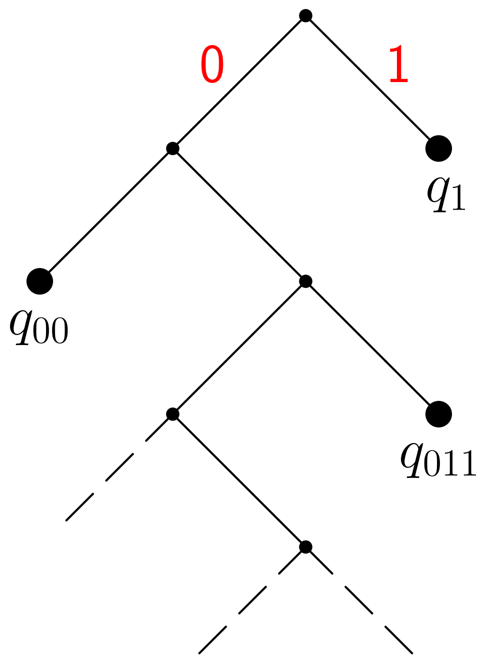


$$U_n = \ldots 1000110$$



$$U_n = \ldots 1000110$$

context : $011 =: \overleftarrow{\text{pref}}(U_n)$



$U_n = \ldots 1000110$

context: $011 =: \overleftarrow{\text{pref}}\,(U_n)$

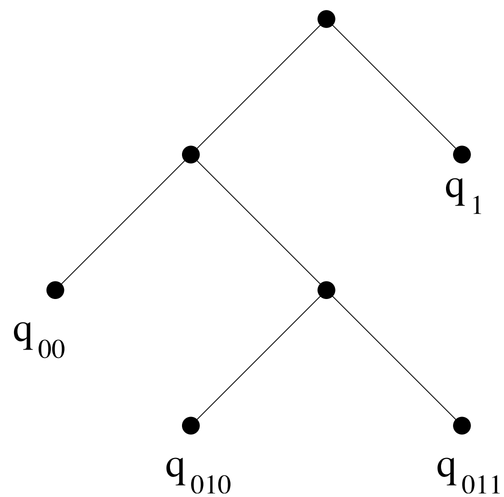
q_{011} = Bernoulli distribution on $\mathcal{A}$

$$\mathbb{P}(U_{n+1} = U_n\alpha | U_n) = q_{011}(\alpha)$$

Definition of the Markov chain:

$$\mathbb{P}(U_{n+1} = U_n\alpha | U_n) = q_{\overleftarrow{\text{pref}}\,(U_n)}(\alpha).$$

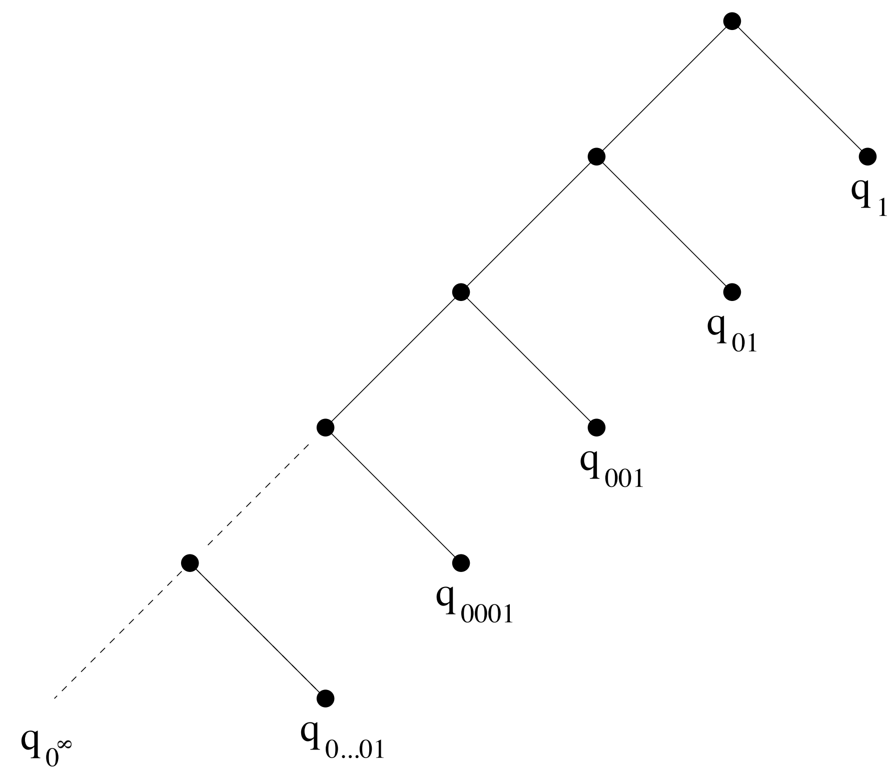
Data: probabilized context tree, with at most countably many infinite branches



finite tree $\rightarrow \quad (X_n)$ Markov chain

infinite tree $\rightarrow \quad (X_n)$ non Markov chain

Question: stationary law for (U_n)?



(X_n) is not Markov.

Stationary measure for the infinite comb

Let (U_n) be a VLMC associated with an infinite comb, probabilized by Bernoulli distributions $q_{0^k 1}, \infty \geq k \geq 0$ on the leaves. Let

$$c_n := \prod_{k=0}^{n-1} q_{0^k 1}(0)$$

Theorem 0.1 *(irreducible case)* $q_{0^\infty}(0) \neq 1$
The Markov process (U_n) admits a stationary measure on $\mathcal{L}$ if and only if the series $\sum c_n$ converges.
In this case, the stationary measure is unique, it is characterized by explicit formulae depending on the $q_{0^k 1}$.

Mixing properties

$\rightarrow$ of a VLMC $(U_n)_{n\in\mathbb{N}}$ defined by an infinite comb where

$$U_n = \ldots X_0 X_1 \ldots X_{n-1} X_n$$

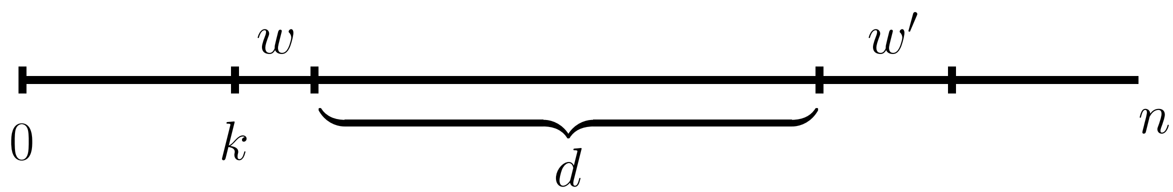
$\rightarrow$ under the stationary distribution π and let

$$\boxed{p_w = \pi(\mathcal{L}w) = \mathbb{P}(X_0 X_1 \ldots X_{|w|-1} = w)}$$

for any finite word w.

Mixing properties of a process

The question: in a random sequence $(X_n)_{n\in\mathbb{N}}$, measure the independence between the occurrences of two finite words w and w' when d letters separate them.



For any $k \geq 0$, compare

$$\mathbb{P}(X_k X_{k+1} \cdots \in w\mathcal{A}^d w' \dots)$$

and

$$\mathbb{P}(X_k \dots X_{k+|w|-1} = w) \times \mathbb{P}(X_{k+|w|+d} \dots X_{k+|w|+d+|w'|-1} = w').$$

In general, take a $\sup_{k\geq 0}$, which is not required when the sequence $(X_n)_n$ is stationary.

For any finite words w and w', consider the function

$$\Phi(d, w, w') = \frac{\left(\sum_{|v|=d} p_{wvw'}\right) - p_w p_{w'}}{p_w p_{w'}}.$$

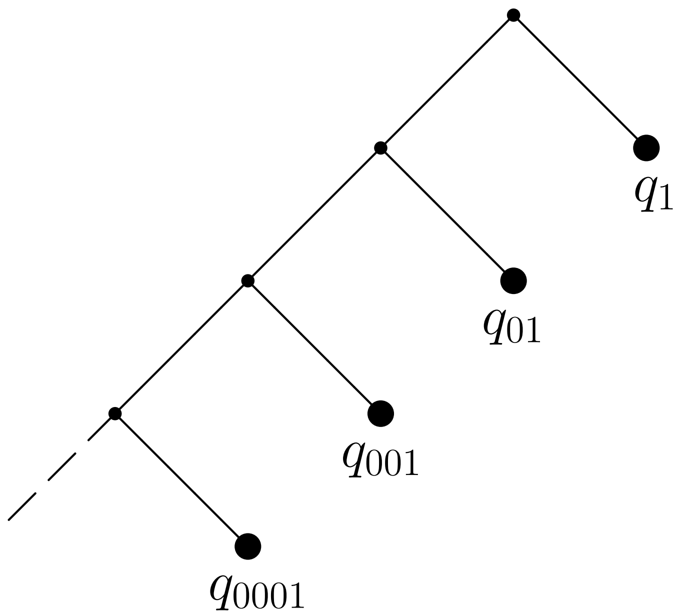
[$\Phi \equiv 0$ when the process $(X_n)_n$ is a sequence of i.i.d. variables]

The mixing property of the process is the decreasing rate to 0 of the function

$$\Phi(d) = \sup_{w,w' \in \mathcal{W}} \Phi(d, w, w').$$

Nice cases: Φ is exponentially decreasing to 0.

Stationary Riemannian infinite comb



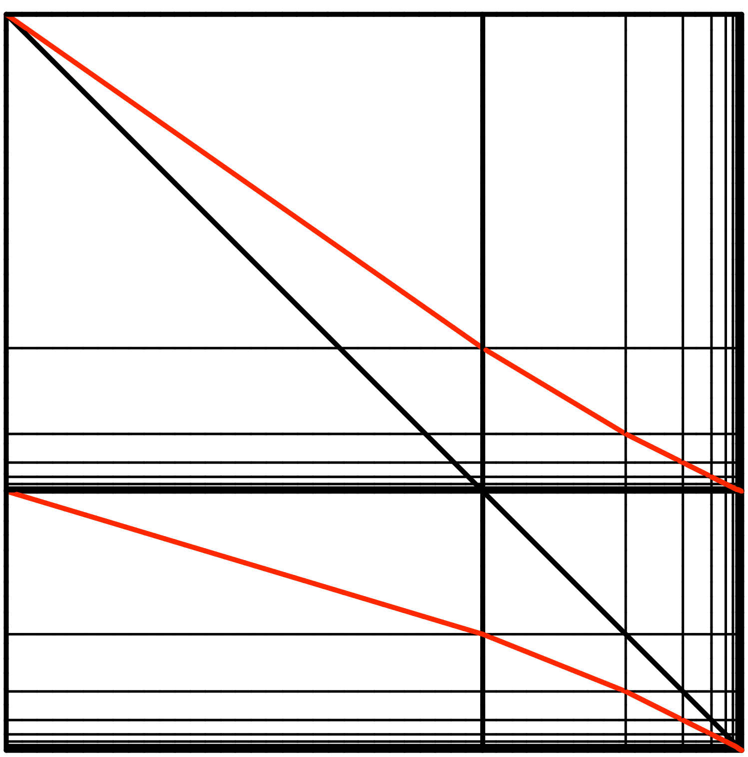
Context probabilities:

$$q_{0^n1}(0) = \left(\frac{n+1}{n+2}\right)^{\alpha}, \ \alpha > 1.$$

$$\boxed{c_n = (1+n)^{-\alpha}}$$

There exists a unique stationary measure π on left-infinite words [CCPP2011].

Consider the VLMC $(U_n)_{n\in\mathbb{N}}$ defined by
- this probabilized context tree;
- the initial distribution $U_0 \sim \pi$.



Mixing of the stationary Riemannian infinite comb

If w and w' are finite words, one gets the explicit formula

$$\Phi(d, w1, 1w') = \zeta(\alpha) u_{d+1} - 1$$

where

$$\sum_{n\geq 0} u_n z^n = \frac{z}{(1-z)\,\mathrm{Li}_\alpha(z)},$$

$\mathrm{Li}_\alpha(z) = \sum_{k\geq 1} \frac{z^k}{k^\alpha}$

Analytic combinatorics!

$$u_n = \frac{1}{\zeta(\alpha)} - \frac{1}{(1-\alpha)\zeta(\alpha)^2}\, n^{1-\alpha} + \mathcal{O}(n^{-\alpha})$$

$$\left|\Phi(d, w1, 1w')\right| \underset{d\to\infty}{\sim} c_\alpha \left(\frac{1}{d}\right)^{\alpha-1}.$$

Suffix Trie of the stationary Riemannian infinite comb

An infinite word $X_0X_1X_2\ldots$ is produced by a stationary Riemannian infinite comb.

Insert in a trie the sequence of the suffixes

$$\begin{aligned} X_0X_1X_2X_3X_4\ldots \\ X_1X_2X_3X_4\ldots \\ X_2X_3X_4\ldots \\ etc\ldots \end{aligned}$$

The height of the trie after the insertion of the n-th suffix is of order $\log n$ under an exponential mixing assumption (Szpankowski, 1983).

Height of a Suffix Trie for a weakly mixing infinite comb

$\rightarrow$ for a "logarithmic" comb (of order 4):

$$\forall \eta > \frac{1}{36}, \qquad a.s. \qquad \frac{L_n}{n^{\frac{1}{4}-\eta}} \longrightarrow +\infty, \qquad n \to +\infty$$

$\rightarrow$ simulations (due to M. Guesdon, INRIA Rocquencourt).

$\rightarrow$ a conjecture:

$$\frac{L_n}{n^{\frac{1}{\alpha}}} \longrightarrow c_\alpha \quad a.s. \text{ or } P$$

Open questions

• get a dictionary between the mixing type of a VLMC and the height of the associated suffix trie; other parameters of the trie?

• NSC of existence and unicity of a stationary measure in more general cases than the infinite comb, depending on the $shape$ of the context tree;

• get a relationship between the renewal properties of some VLMC and the spectral properties of the transfer operator associated with the dynamical system;

• approximate a VLMC by a sequence of Markov chains of increasing order;

• is it possible to see a VLMC as a process on the contexts (automaton)?