Optimal Selection via Dynamic Programming

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Numerous improvements; some recent. However:

The optimality of these algorithms is asymptotic.

None is acceptable for small sets.

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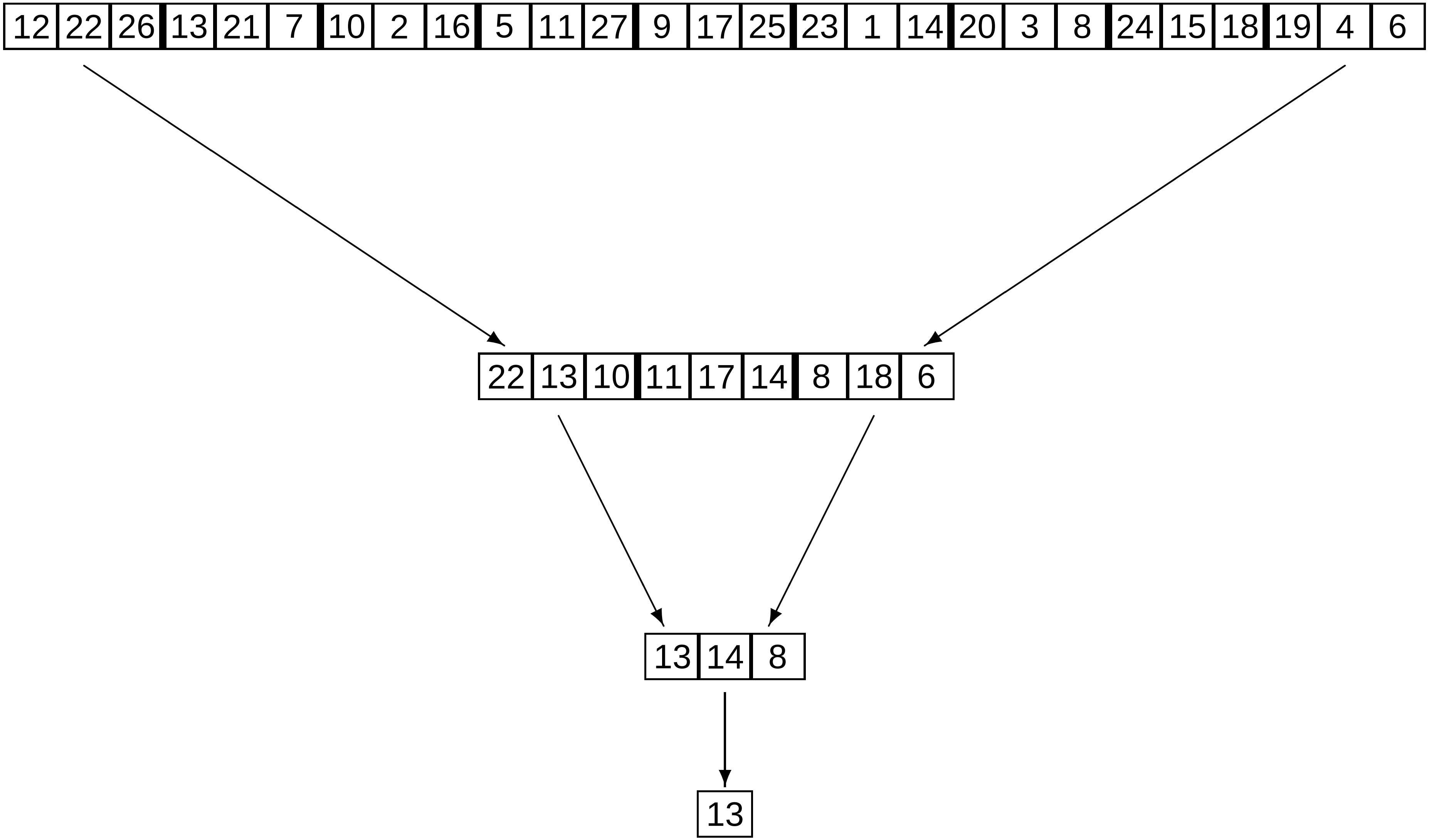
The algorithm is deterministic, and one of its versions uses at most $1.5n$ comparisons,

The expected number is $4/3n$.

Extremely efficient, and easy to implement;

but it only approximates the median.

Example


12 22 26 13 21 7 10 2 16 5 11 27 9 17 25 23 1 14 20 3 8 24 15 18 19 4 6
22 13 10 11 17 14 8 18 6
13 14 8
13

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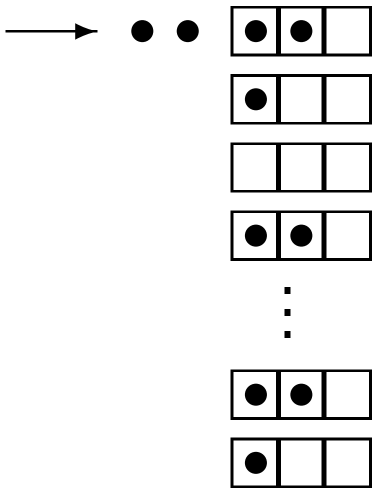
Essentially the same algorithm can be done "on-line," processing streaming data.



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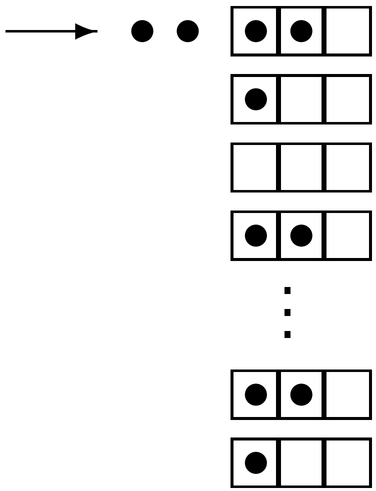

For n entries we need work-area of $4\log_3 n$ positions;

or $(b+1)\log_b n$ when using buffers of size b.

Cost of search – in comparisons

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The expected total number of comparisons when looking

in a list of size n is $\boxed{C_b n,}$ where $C_b = \overline{V}_m(b)/(b-1)$

b	3	5	7	9
$\overline{V}_m(b)$	2.667	5.867	9.305	12.790
$V_m(b)$	3	6	10	14
C_b	1.333	1.4667	1.3293	1.5987

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Here is why: Say $n = b^r$. We need $n/b,\ n/b^2, \ldots n/b^r = 1$

exact medians. The expected cost is

$$n\overline{V}_m(b) \sum_{j=1}^{r} b^{-j} \approx n\overline{V}_m(b)/(b-1) \stackrel{\text{def}}{=} nC_b$$

Searching for a selection algorithm

How to do it? How to locate fast—on the average—the median for $b = 5, 7, 9$? nothing much that I could find, except. . .

Knuth provides a table with these numbers, no algorithm or proof, nothing about derivation; but he has some insights:

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Formal derivation needs notation:

S – the information state about the entries, a digraph.
Edge from large to small.

$U(S)$ – the expected cost to complete the selection, starting at state S.

$\mathscr{S}_r$ – the set of "resolved" states, which hold enough information to determine the selection.

$C(S)$ – set of possible actions at state S.

Searching for a selection algorithm

With this notation we have the natural recurrence (MDP)

$$U(S) = 1 + \min_{\varepsilon \in C(S)} E_\varepsilon U(S \mid \varepsilon), \qquad S \notin \mathscr{S}_r, \tag{1}$$

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This is a Bellman optimality equation. Satisfies the conditions for the final value $U(S_0)$ to be the optimal cost of resolving S_0.

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This recurrence is more versatile than may appear.

Optimality in worst-case behavior

If we change the operator E_{ε} for $\max_{\varepsilon}$ we see,

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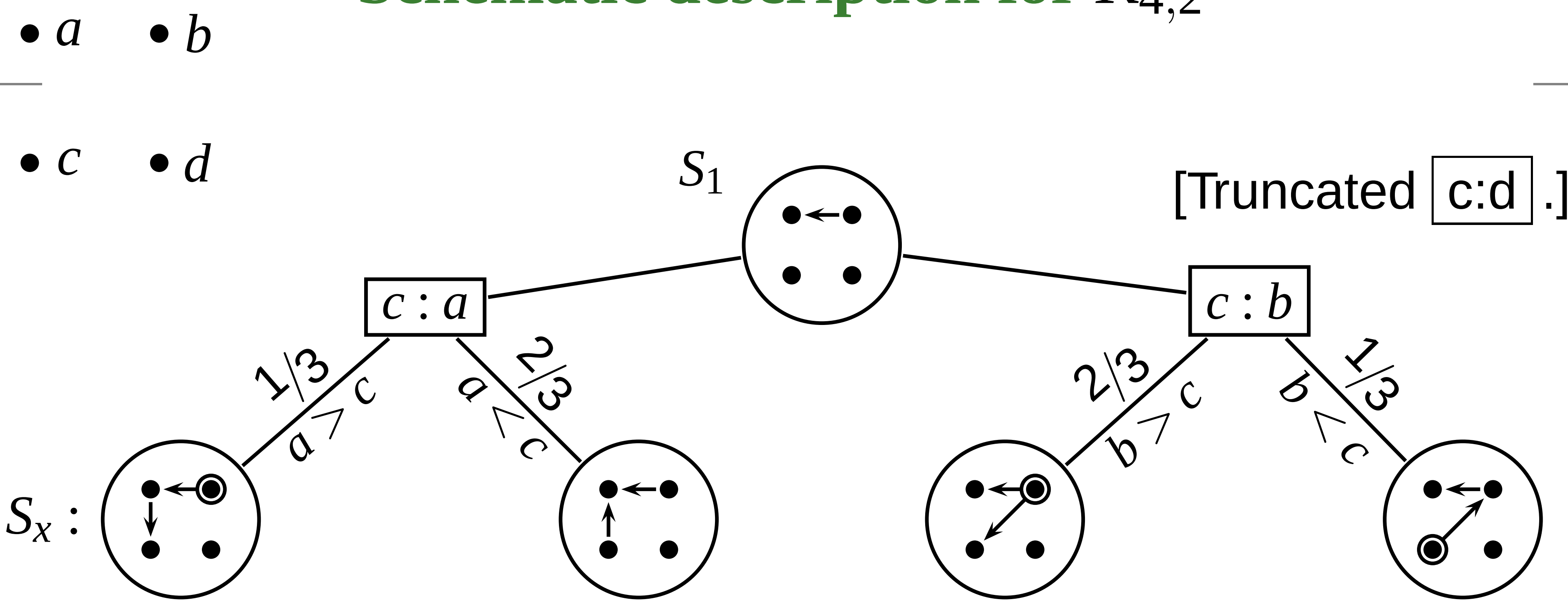
Difficulty: Billions-and-billions to choose from!

$$u_n = 3^{n(n-1)/2}; \qquad u_8 = 2.28768 \times 10^{13}$$

Schematic description for $K_{4,2}$



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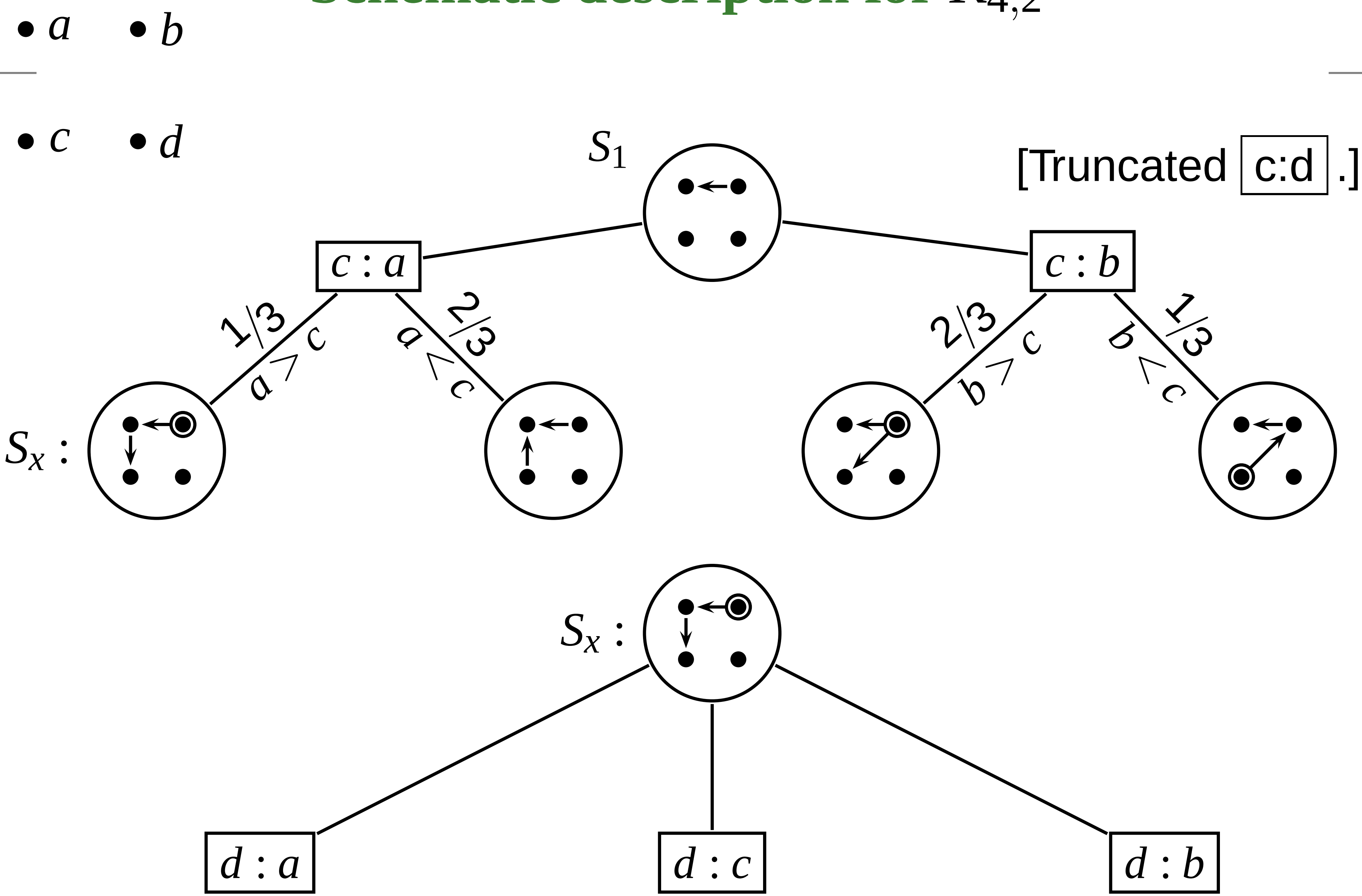

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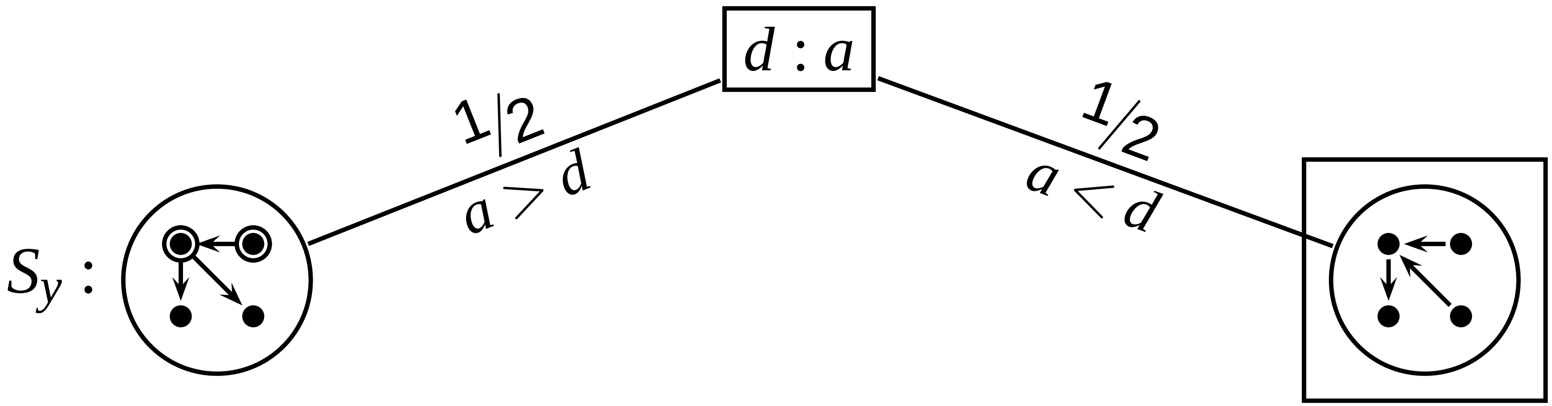

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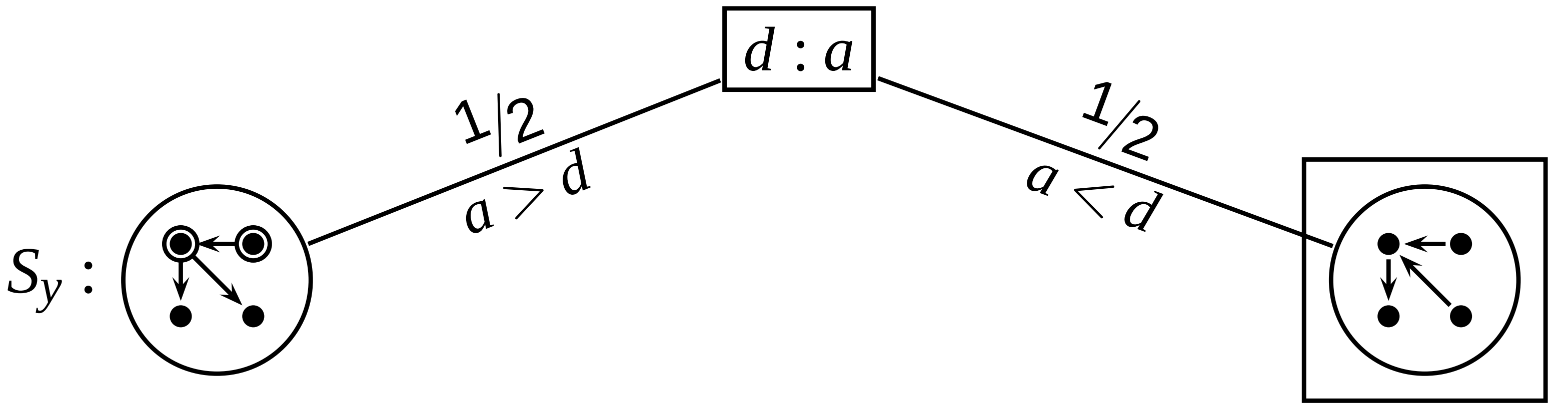

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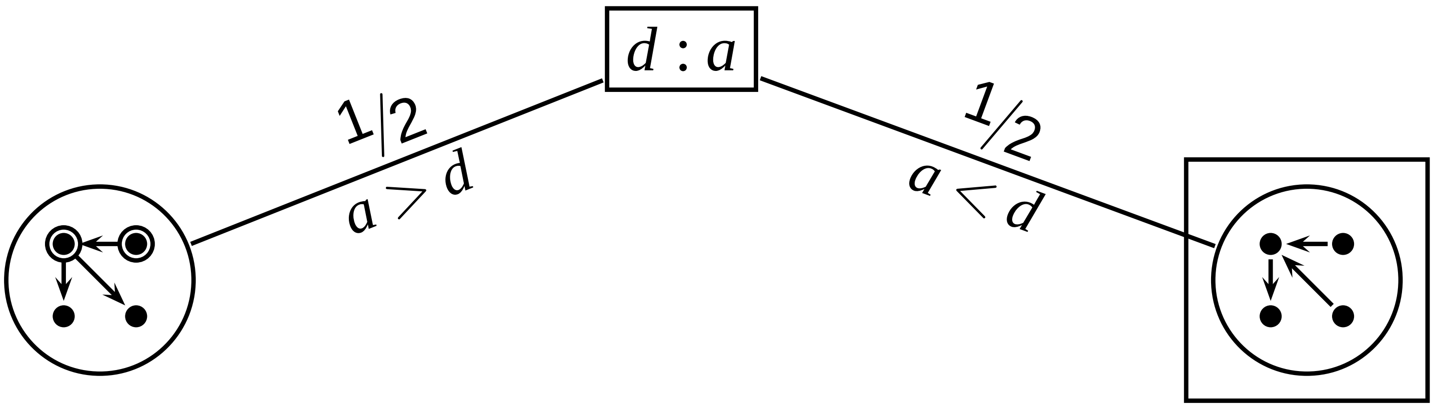

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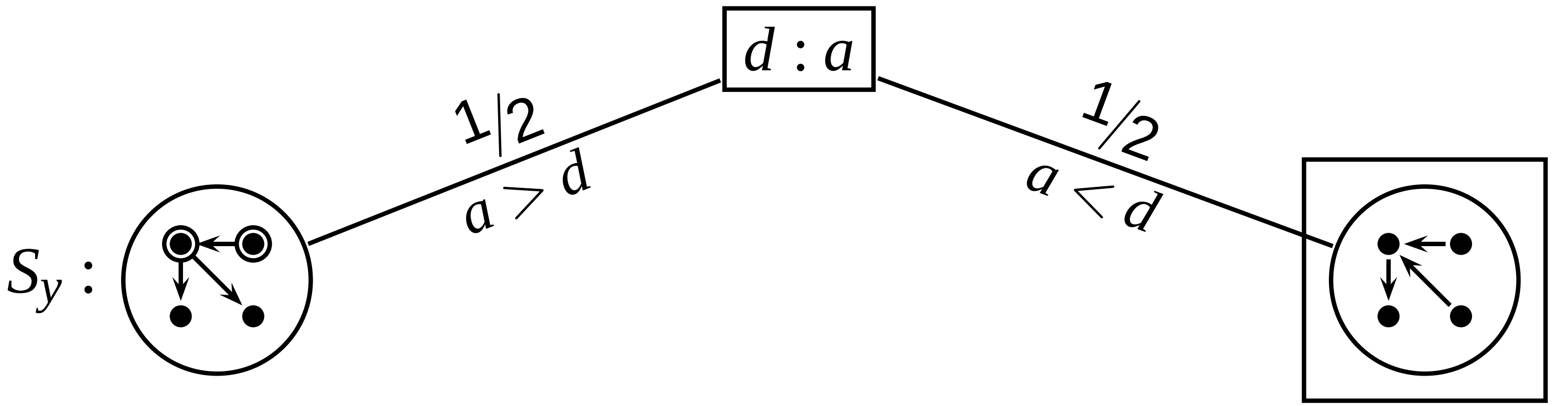

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A general rule: exclude entries known to be too extreme.

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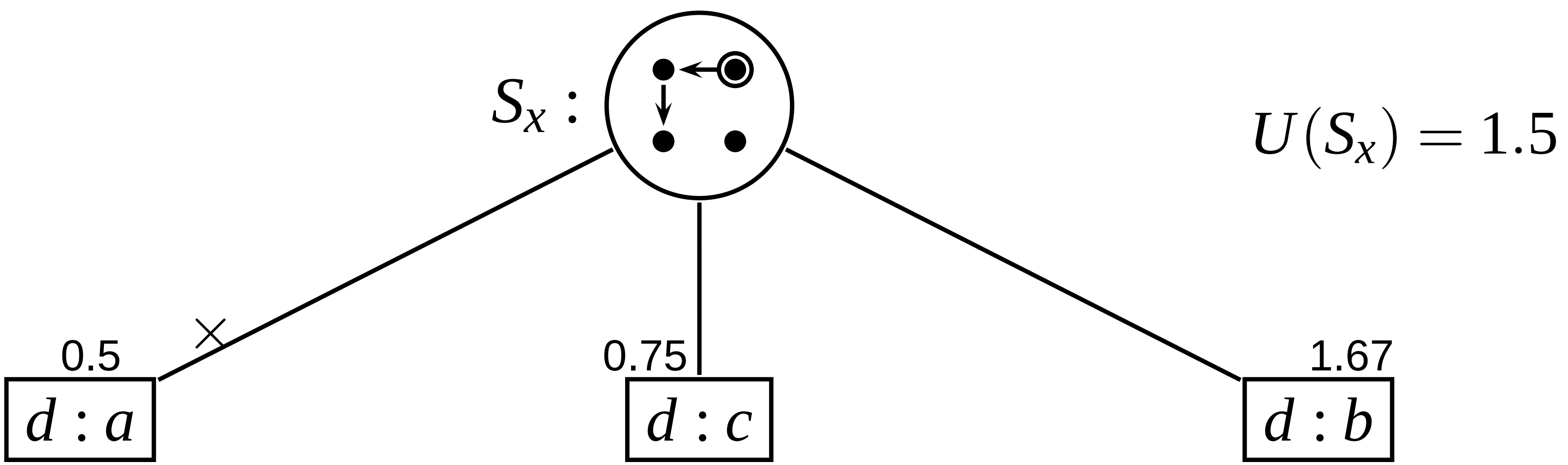

Computational process, Cont.

(4). A resolved state is marked with the item selected.

(5). Now an entire optimal policy can be captured.

Here is a function to locate $K_{4,2}$, as generated by the search:

```
int K42(int *Q) {
if (Q[0] > Q[1]) swap(&Q[0], &Q[1]);  // symmetrization;
L0: if(Q[2] > Q[3]) goto L1;  else goto L2;
L1: if(Q[0] > Q[3]) goto L3;  else goto L4;
L2: if(Q[0] > Q[2]) goto L5;  else goto L6;
L3: if(Q[0] > Q[2])  return Q[2];  else return Q[0];
L4: if(Q[3] > Q[1])  return Q[1];  else return Q[3];
L5: if(Q[0] > Q[3])  return Q[3];  else return Q[0];
L6: if(Q[2] > Q[1])  return Q[1];  else return Q[2];
 }
```

This looks like a full, perfect binary tree. Same cost (4) on all paths. Highly unusual.

Note: the states corresponding to L1 and L2 are isomorphic, and so are the four last states, L3 to L6.

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n	4	5	6	7	8	9
d_n	219	4231	130,023	6,129,859	431,723,379	44,511,042,511
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The largest we did was $n = 11$:

$$d_{11} = 1,396,281,677,105,899; \qquad d_{11}^* = 46,749,427.$$

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n	$b = 3$	$b = 5$	$b = 7$
10	0.5215	0.6913	0
100	5.2242	4.2694	5.5255
1000	26.101	17.999	31.838
10,000	144.19	83.239	92.766
100,000	648.86	412.33	310.71
1,000,000	3471.2	2801.1	1297.9
10,000,000	11901	5787.7	4782.8