INFORMATION THEORY: SOURCES and DIRICHLET SERIES, TAMENESS of DYNAMICAL SOURCES

Brigitte Vallée

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(CNRS and University of Caen, France)

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Talk partly based on joint works with

Viviane Baladi, Eda Cesaratto,

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Dedicated to Philippe

Plan of the talk.

Part I

– describes a general model for sources

– shows the importance of the Dirichlet generating functions

– explains the importance of tameness in the analyses of text algorithms

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Part II

– defines a natural subclass of sources, the dynamical sources

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Part II

– defines a natural subclass of sources, the dynamical sources

Part III

– provides sufficient conditions for tameness of dynamical sources

Part I

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– shows the importance of the Dirichlet generating functions

– explains the importance of tameness in the analyses of text algorithms

Sources.

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one symbol for each time unit.
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Fundamental role of the Dirichlet generating functions of the source

$$\Lambda(s) := \sum_{w \in \Sigma^\star} p_w^s, \qquad \Lambda_k(s) = \sum_{w \in \Sigma^k} p_w^s$$

Remark: $\Lambda_k(1) = 1$ for any k, $\qquad \Lambda(1) = \infty$.

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– the entropy $h_{\mathcal{S}}$,

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– the coincidence $c_{\mathcal{S}}$

$c_{\mathcal{S}}(A, B) :=$ the length of the longest common prefix of A and B

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– they intervene in the realistic analyses of sorting and searching algorithms

Text algorithms and dictionaries : The trie structure

An example : A trie built on a set of 16 words.

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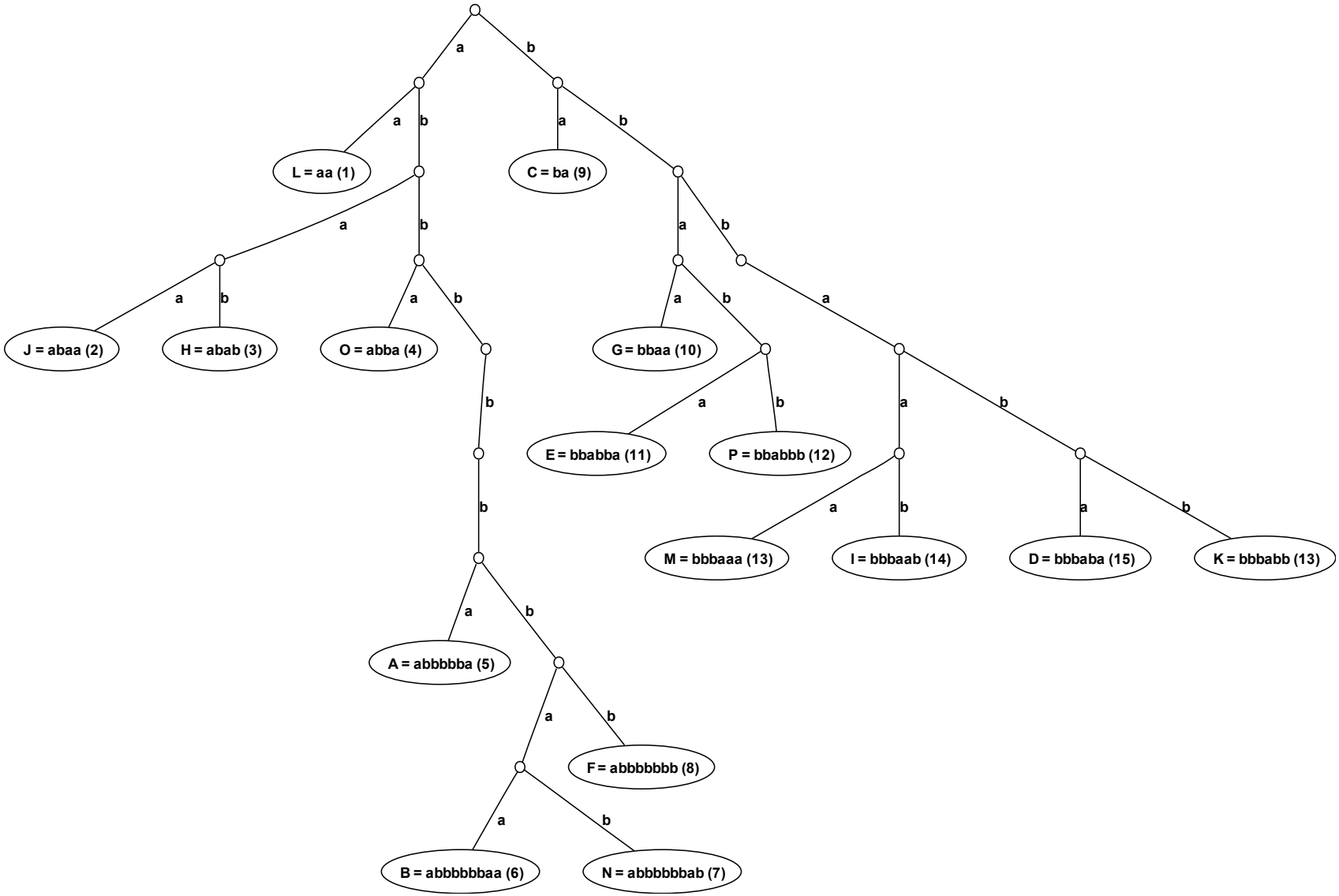

Probabilistic study of the `Trie` structure.

Main parameter on a node n_w labelled with prefix w:

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Role of $p_w :=$ the probability that a word begins with prefix w.

Realistic probabilistic study of the BST (binary search tree) built on words

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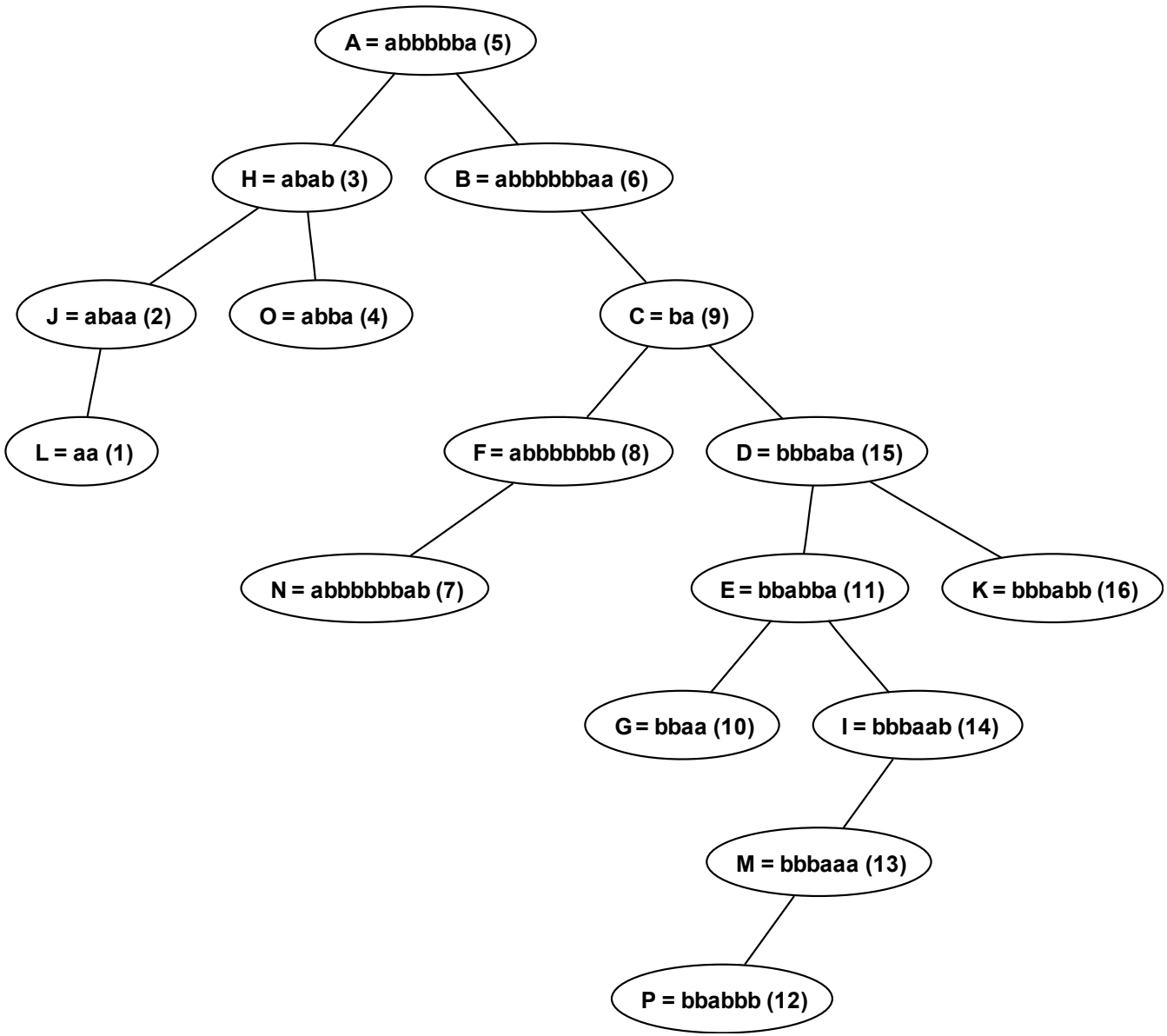

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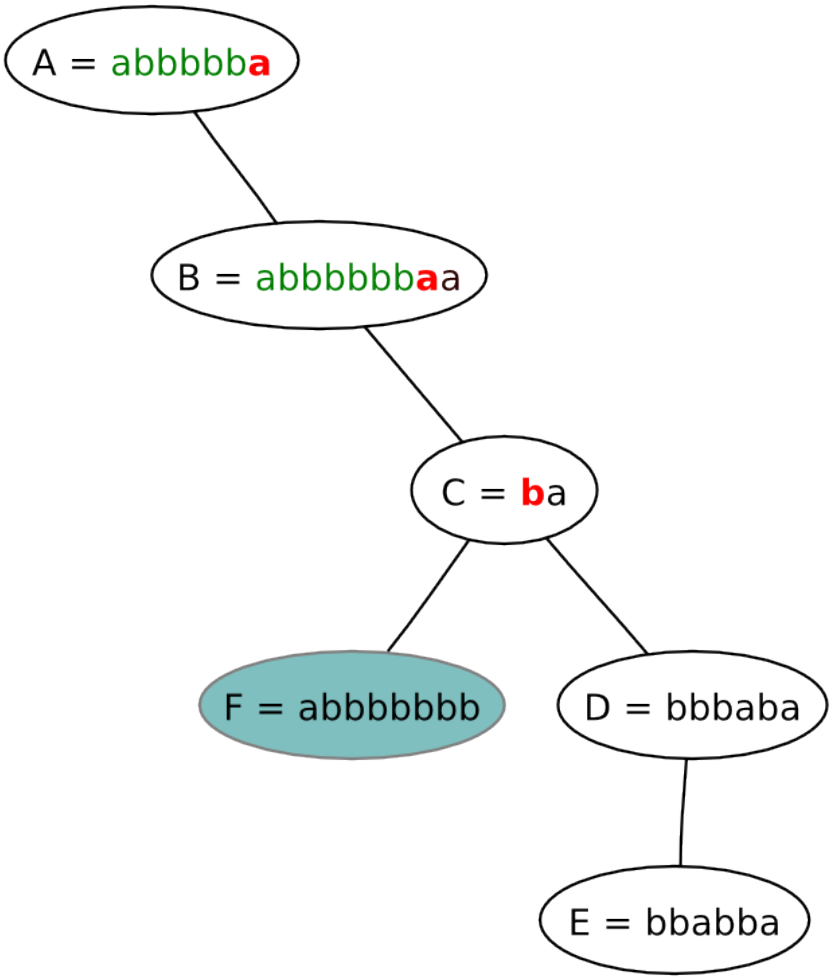

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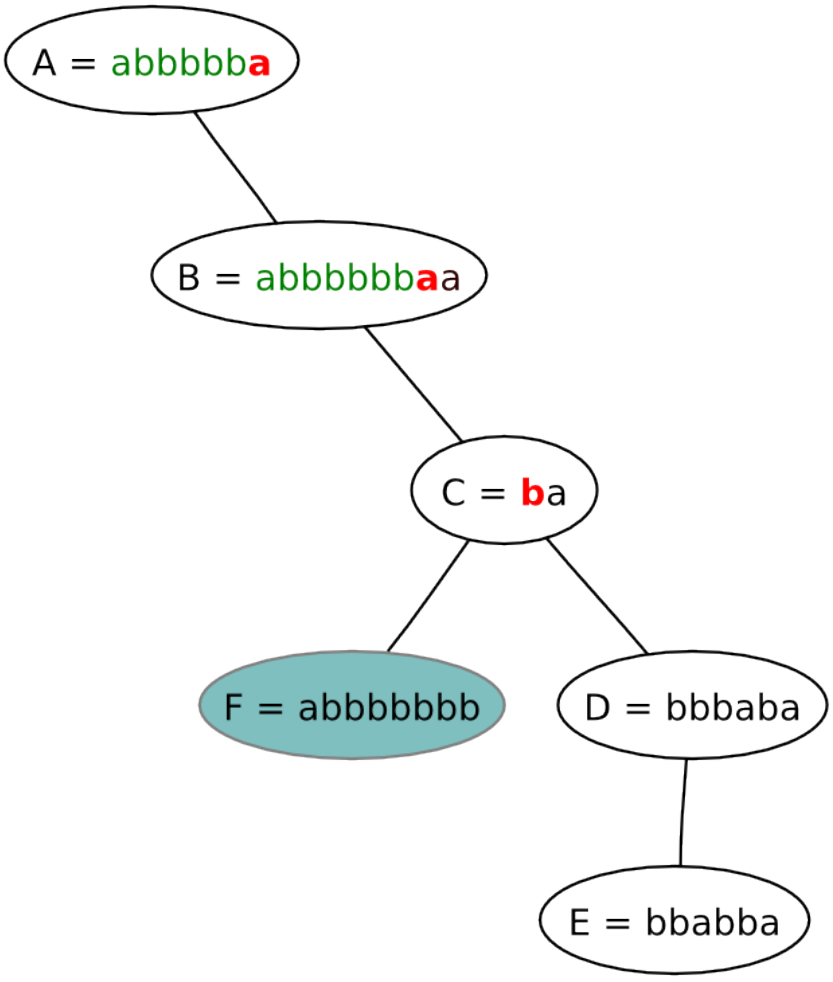

The realistic cost of the comparison between two words A and B is related to their coincidence $c(A, B)$

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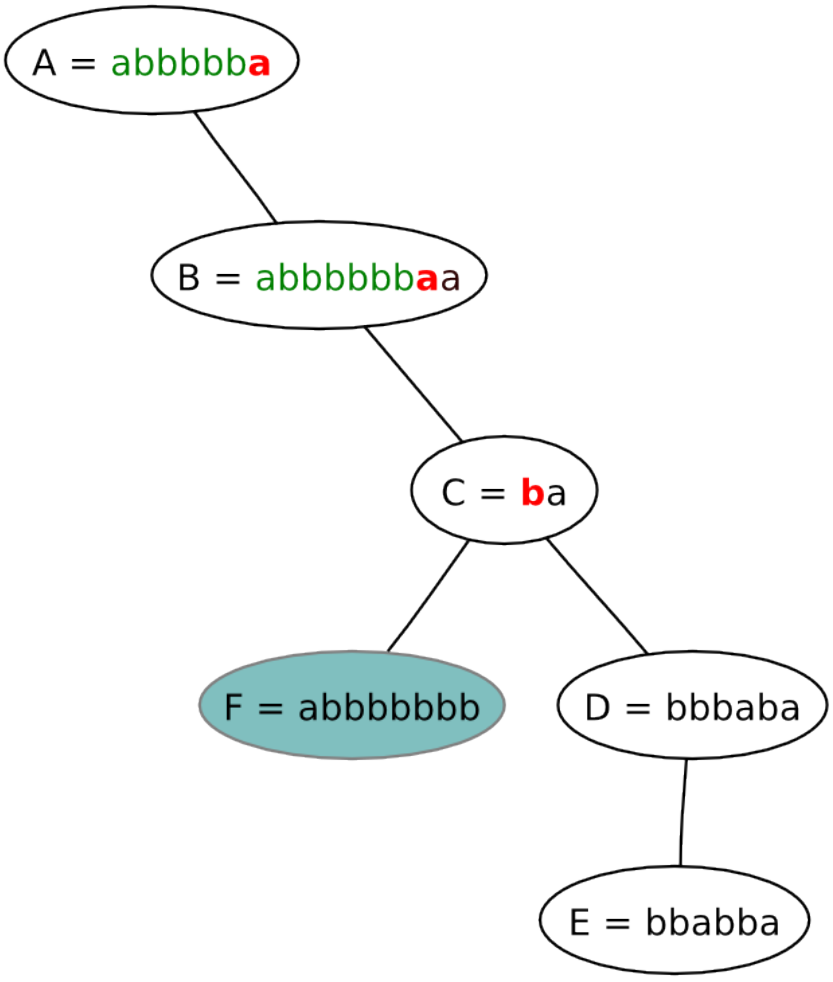

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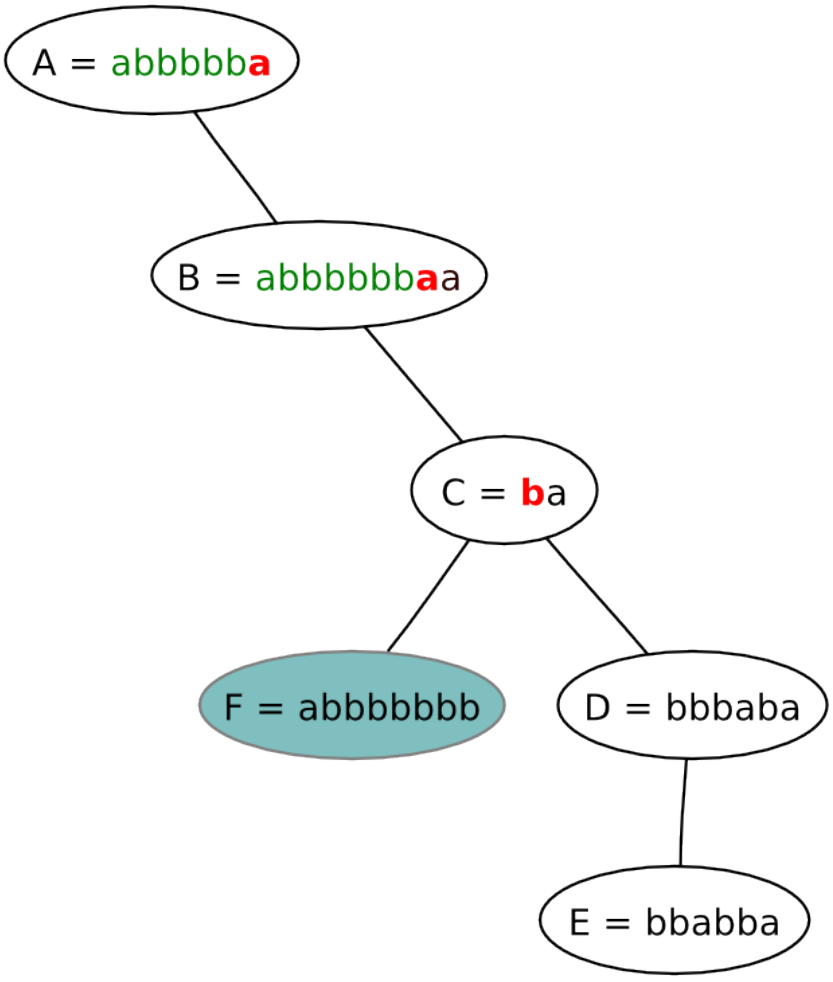

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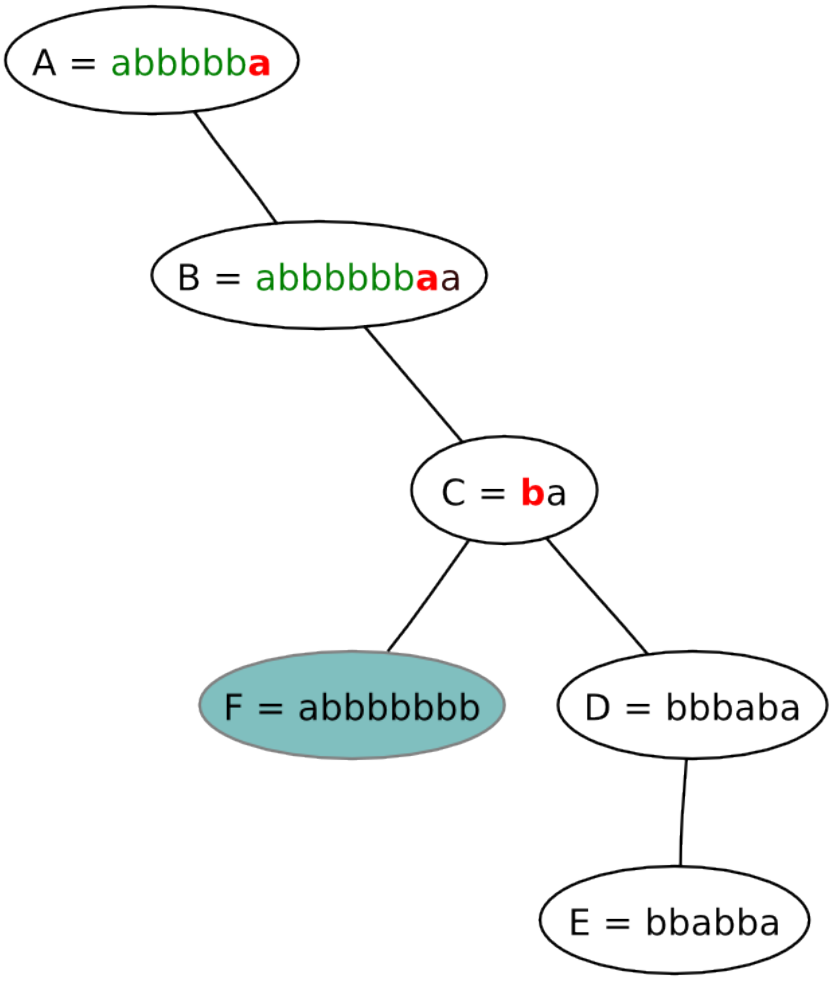

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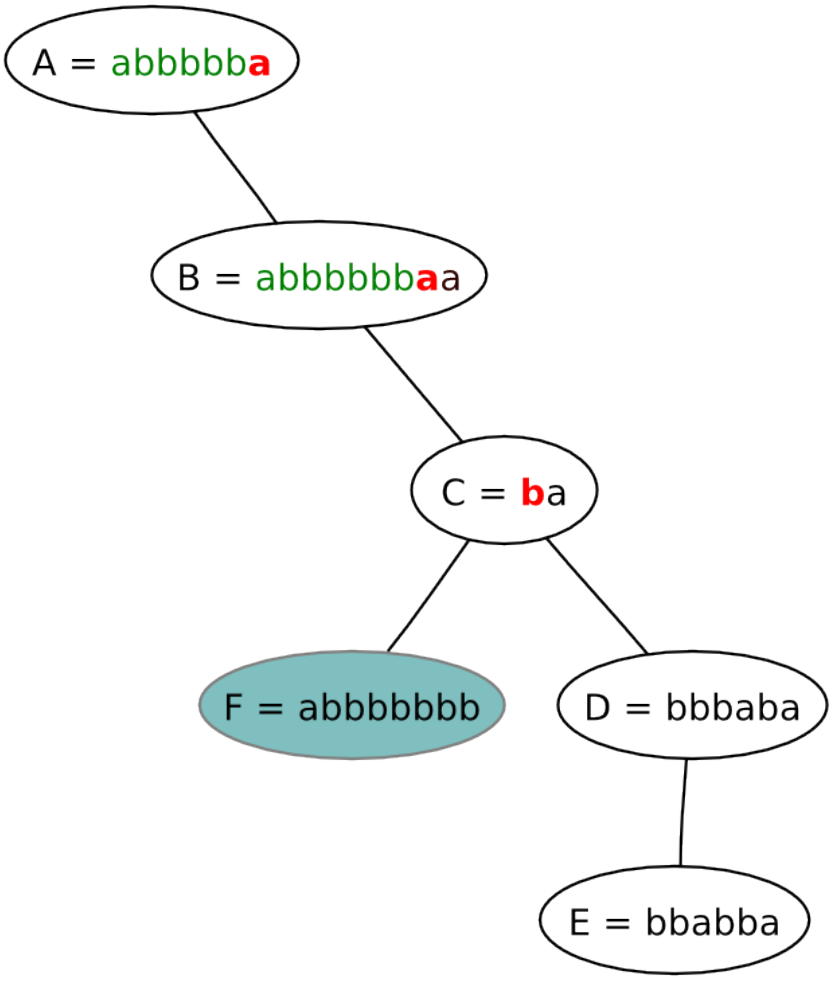

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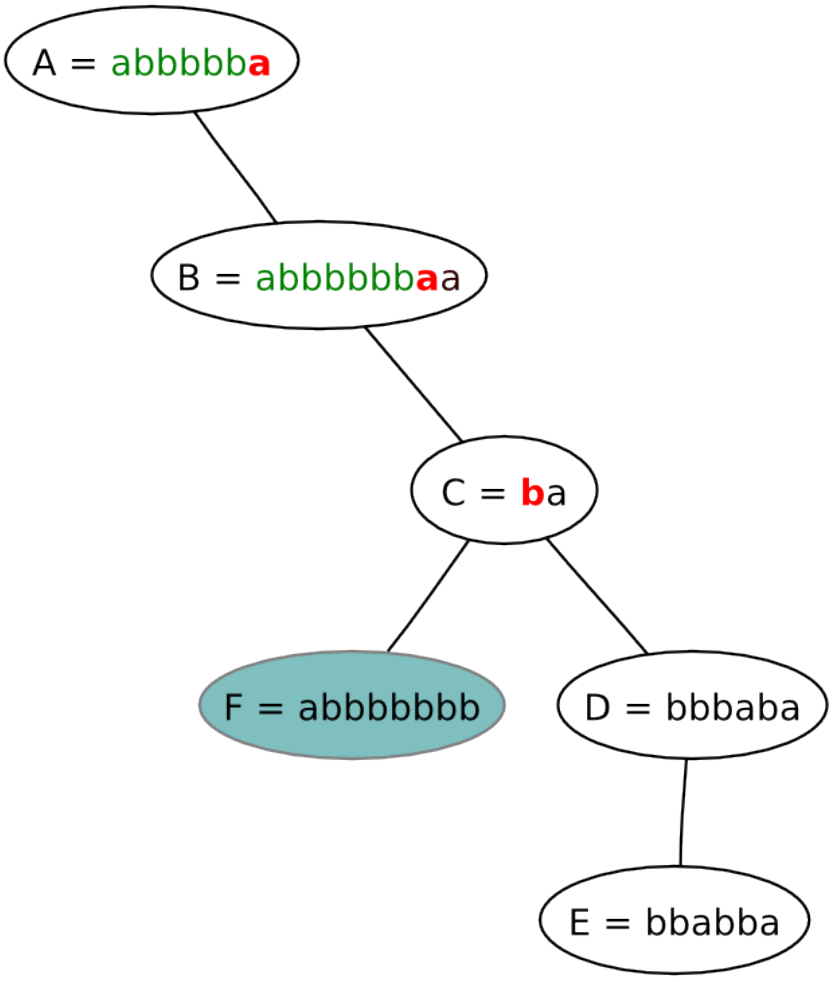

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Exact average-case analysis for Tries or BST's

$S_n^{(X)}$:= the mean path-length for the Trie $[X = T]$

or the mean symbol path-length of the BST $[X = B]$

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$$S_n^{(X)} = \sum_{k=2}^{n} (-1)^k \binom{n}{k} \varpi_X(k)$$

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Nice exact formula, not easy to deal with, due to the alternating signs

Asymptotic analysis.

The residue formula transforms the sum into an integral with $1 < d < 2$.

$$S_n = \sum_{k=2}^{n} (-1)^k \binom{n}{k} \varpi(k) = \frac{1}{2i\pi} \int_{d-i\infty}^{d+i\infty} \varpi(s) \frac{n!\,(-1)^{n+1}}{s(s-1)\dots(s-n)} ds,$$

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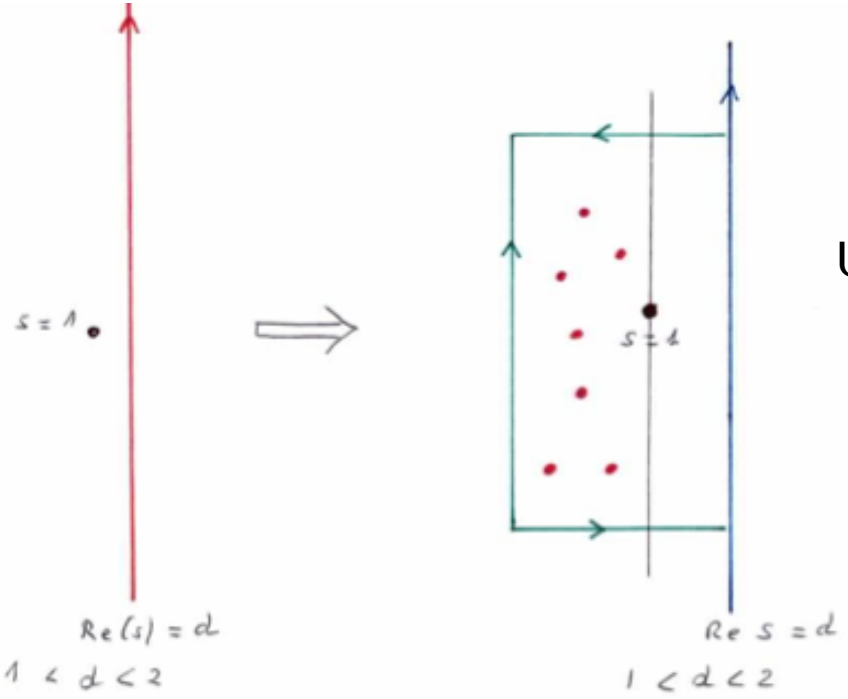

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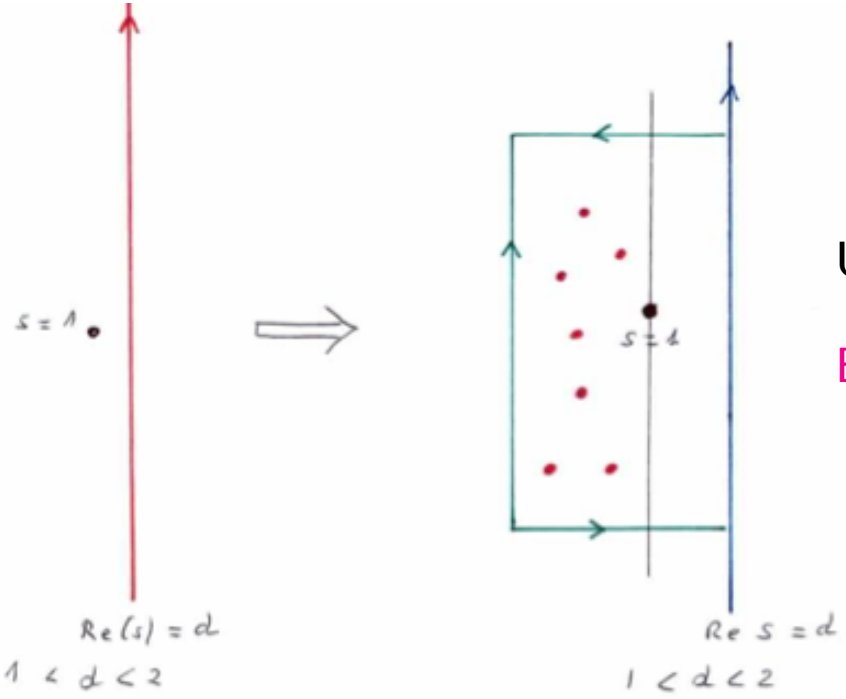

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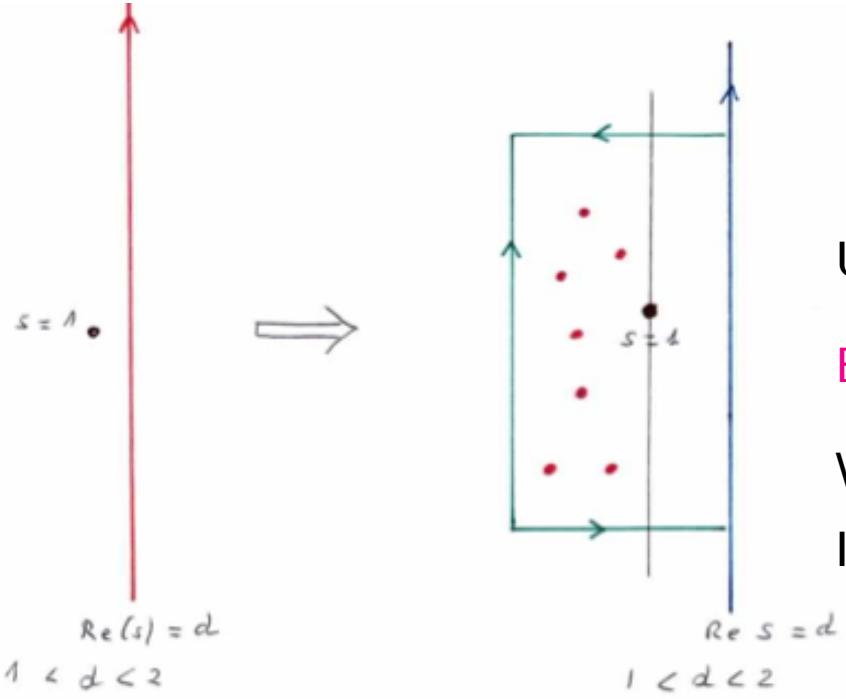

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Is $\Lambda(s)$ of polynomial growth on the green contour?

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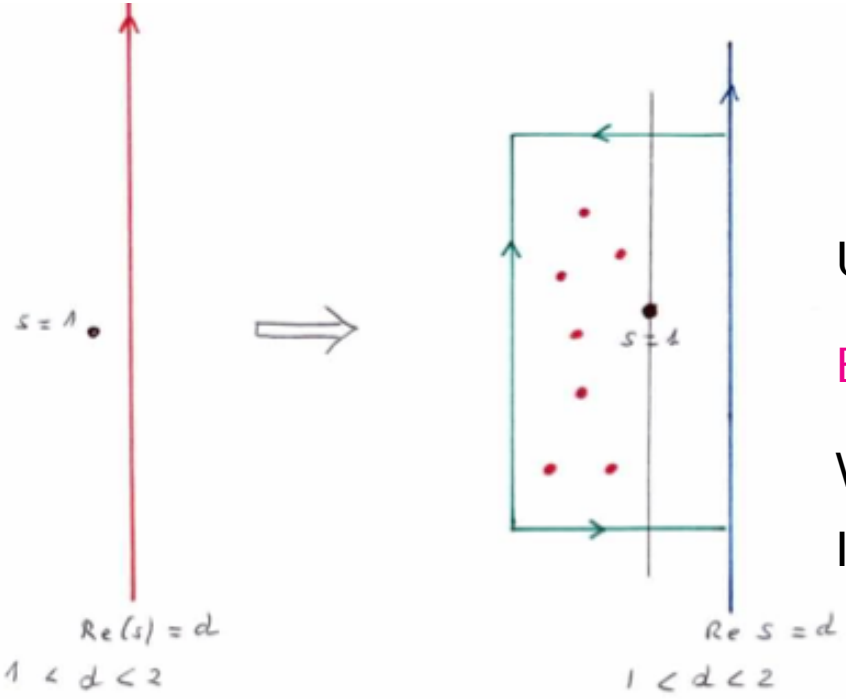

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Importance of the existence of a region $\mathcal{R}$

– which contains only $s = 1$ as a pole – where $\Lambda(s)$ is of polynomial growth.

Tameness of the source

Part II

– defines a natural subclass of sources, the dynamical sources

A general class of "natural" sources: dynamical sources associated to dynamical systems [V. 2001]

With a shift map $T : \mathcal{I} \to \mathcal{I}$ and an encoding map $\sigma : \mathcal{I} \to \Sigma$,
the emitted word is $M(x) = (\sigma x, \sigma T x, \sigma T^2 x, \ldots \sigma T^k x, \ldots)$
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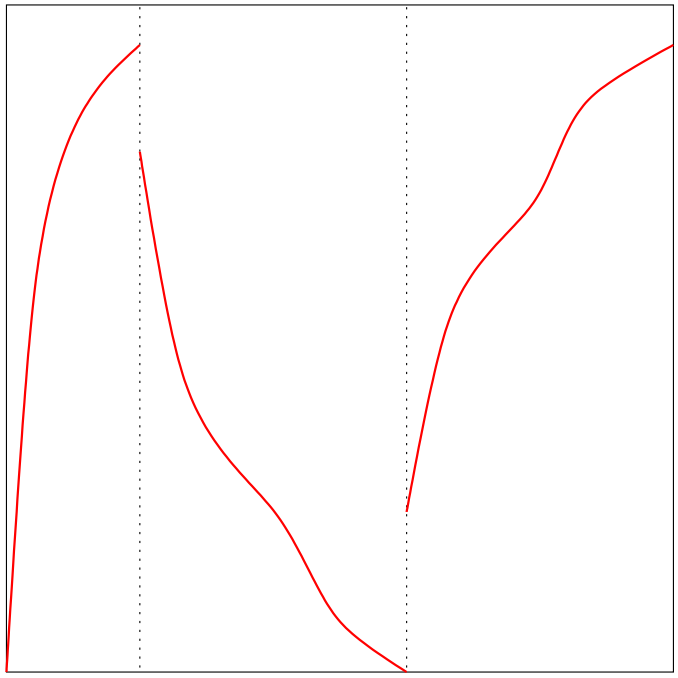

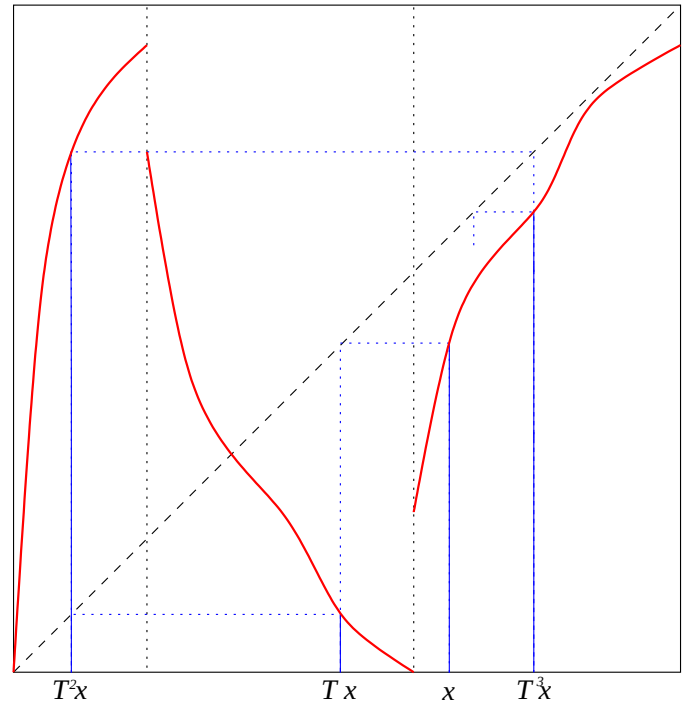

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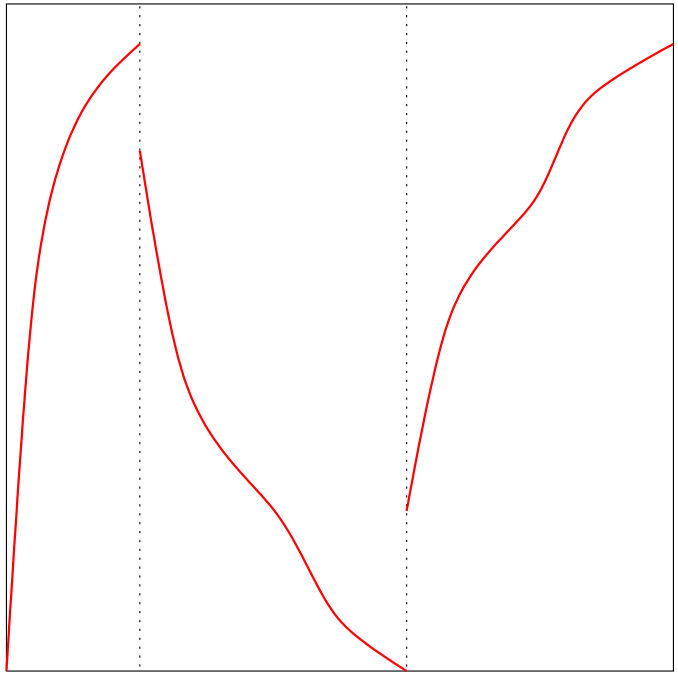

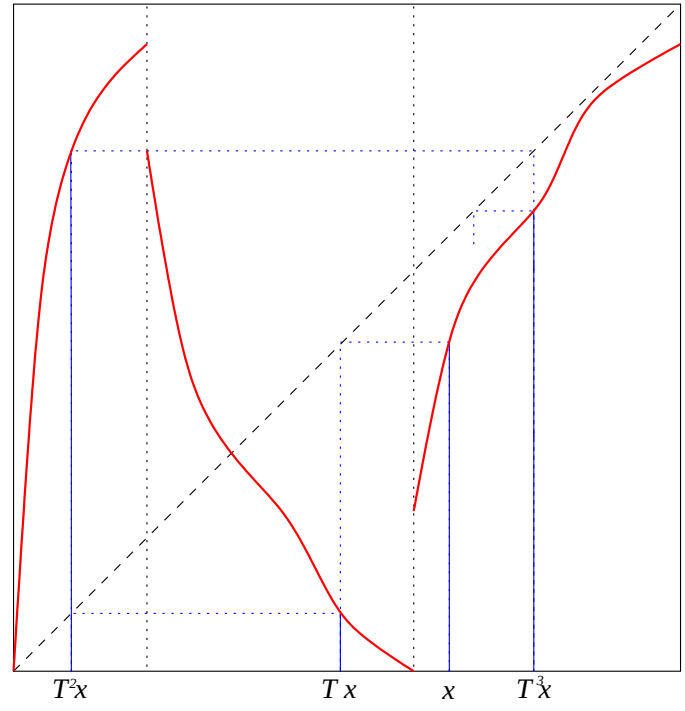

A dynamical system, with $\Sigma = \{a, b, c\}$ and a word $M(x) = (c, b, a, c \ldots)$.

A dynamical source $=$ a source built with a dynamical system

A dynamical system $(\mathcal{I}, S)$ is defined by four elements:

– a finite alphabet Σ,

– a topological partition of $\mathcal{I} :=]0,1[$ with open intervals $\mathcal{I}_{m, m\in\Sigma}$,

– an encoding mapping σ equal to m on each $\mathcal{I}_m$,

– a shift mapping T

s.t. $T|_{\mathcal{I}_m}$ is a bijection of class $\mathcal{C}^2$ from $\mathcal{I}_m$ to $\mathcal{J}_m := T(\mathcal{I}_m)$.

The local inverse of $T|_{\mathcal{I}_m}$ is denoted by h_m.

This gives rise to a source: on an input x of $\mathcal{I}$, it outputs the word

$$M(x) := (\sigma x, \sigma T x, \sigma T^2 x, \dots).$$

When an initial density –and an initial distribution F– is chosen on $\mathcal{I}$,

this induces (via M) a probabilistic model on Σ^∞

$=$ a dynamical source $\mathcal{S}_F$.

Strong relations between the geometry of the system
and the probabilistic properties of the source.

Correlations between symbols are mainly due to two geometric characteristics:
the position of the branches and the shape of the branches.

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Particular cases: – Complete systems $T(\mathcal{I}_m) = \mathcal{I}$
– Markovian systems $T(\mathcal{I}_m) = \text{union of some } \mathcal{I}_\ell$

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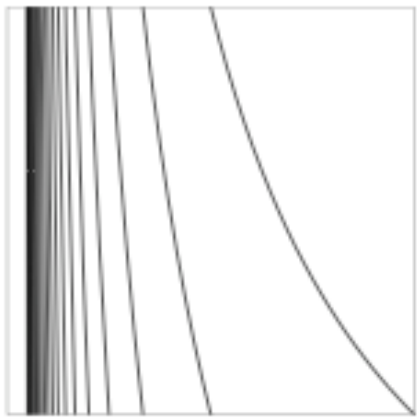
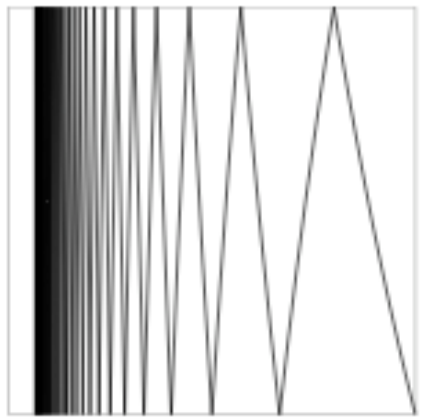
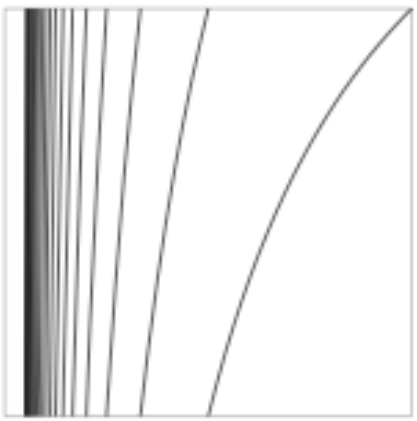
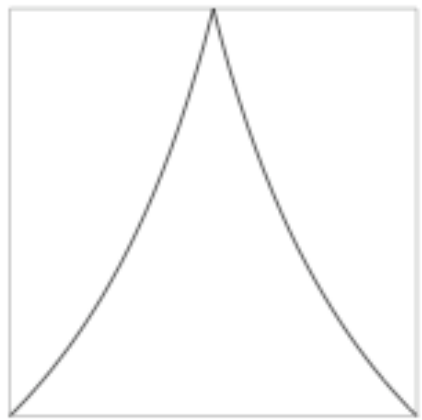
Generally speaking, importance of expansiveness:
the derivative T' satisfies $\forall x \in \mathcal{I} \quad |T'(x)| \geq \delta > 1$.

Four Euclidean dynamical sources, Two different behaviours for trajectories.









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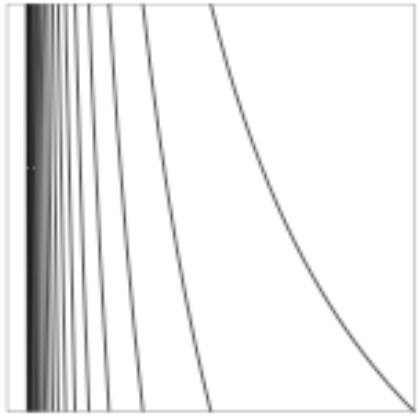

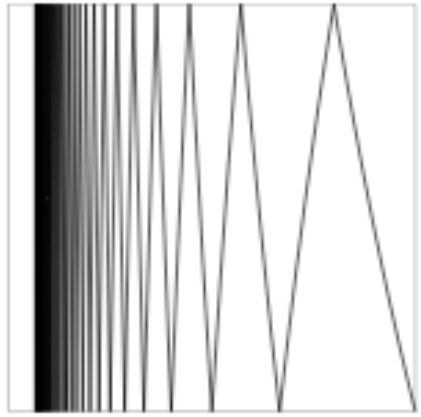

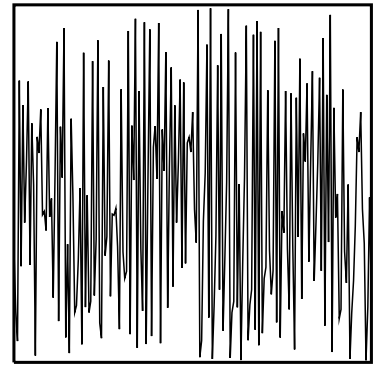

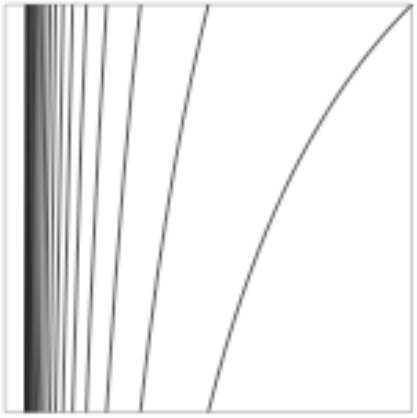

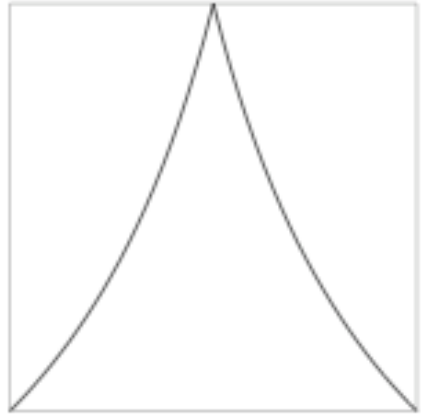

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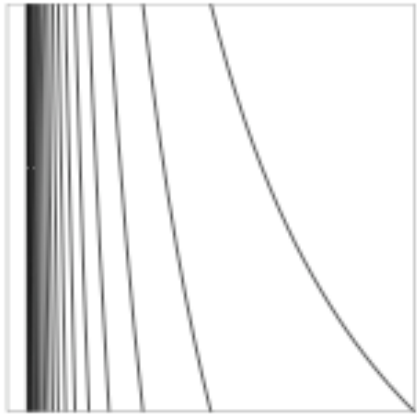

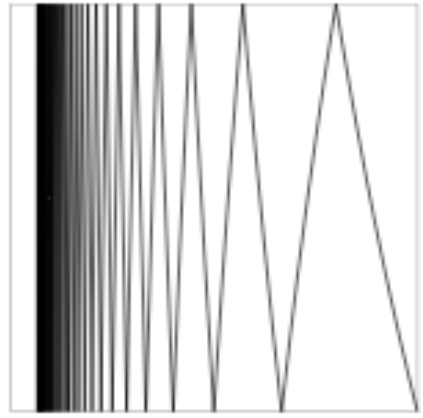

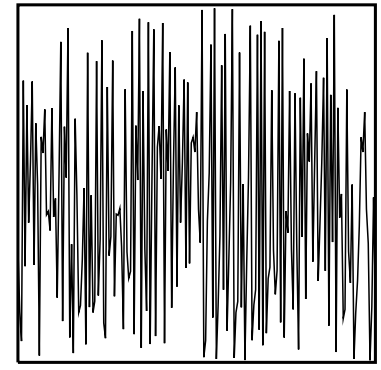

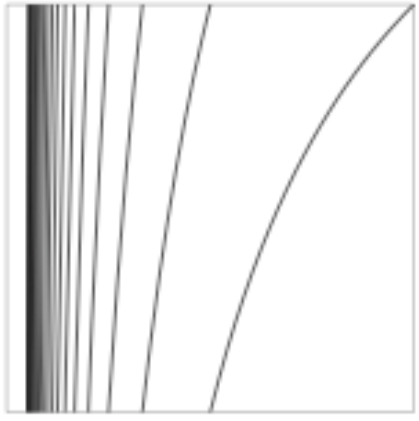

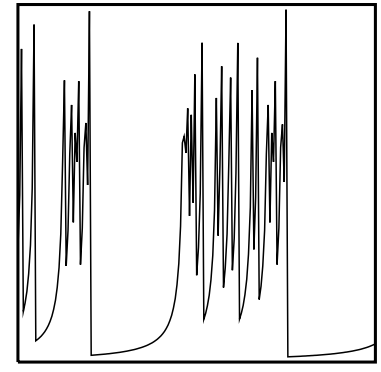

Particular cases: simple sources and affine branches

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:= a complete system with affine branches and uniform initial density

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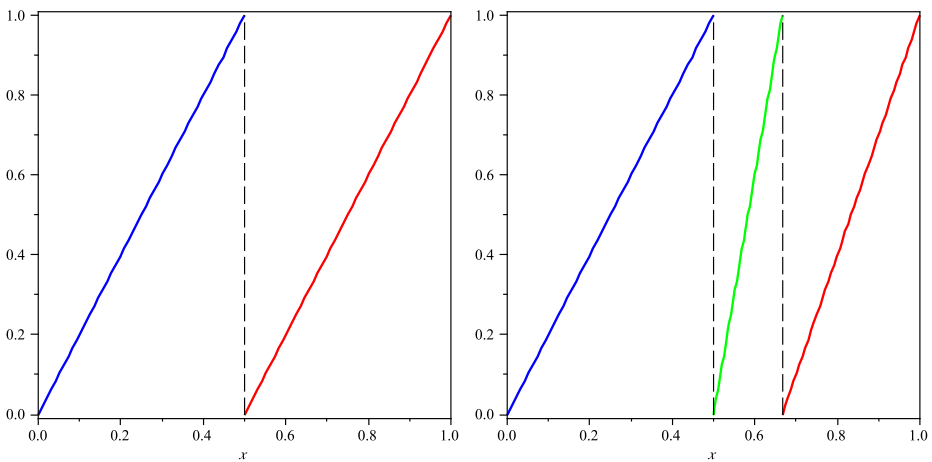

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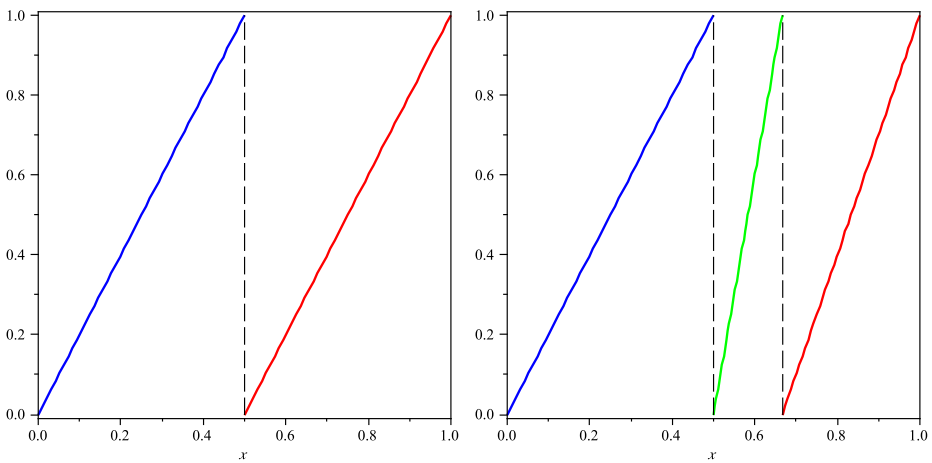

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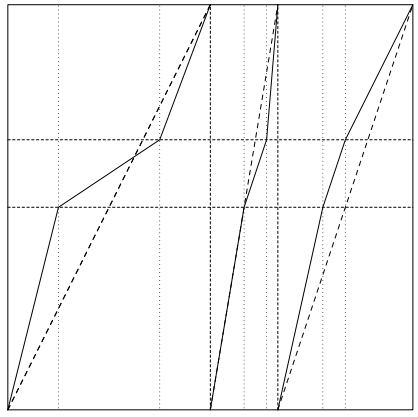

a Markov chain.

General case of interest: the Good Class

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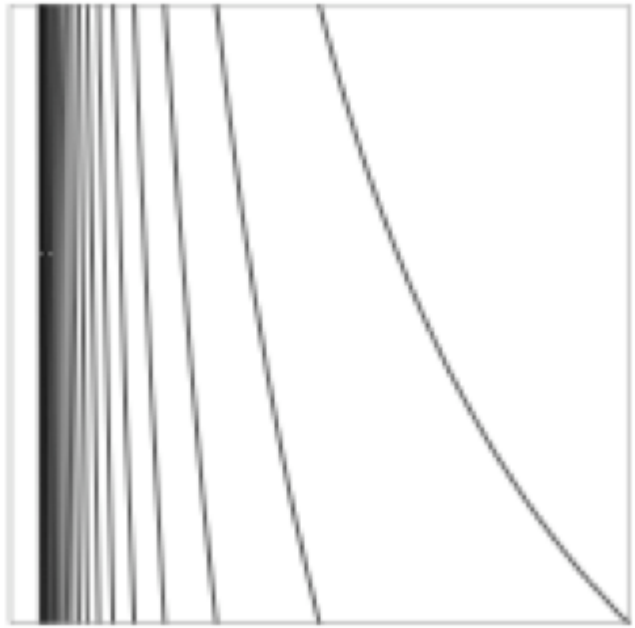
Main instance: the Euclidean source defined with $T(x) := \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$



Expression of the Dirichlet series of the source $\Lambda(s) := \sum_{w \in \Sigma^\star} p_w^s$

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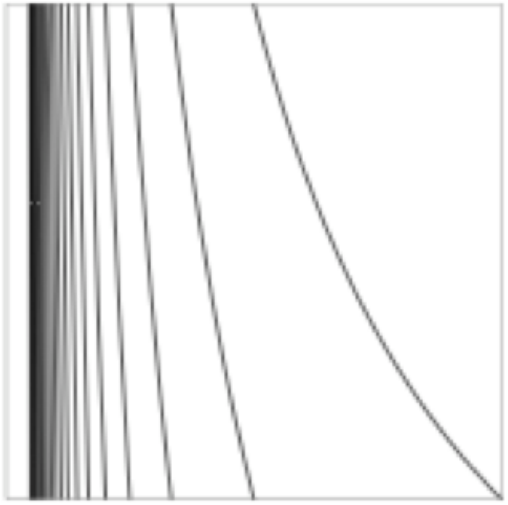
A general dynamical source

$\Lambda(s)$ closely related to $(I - \mathbb{H}_s)^{-1}$

where $\mathbb{H}_s$ is the (secant) transfer operator of the dynamical system.

The density transformer and the transfer operators



The operator $\mathbf{H} := \sum_{m \in \Sigma} \mathbf{H}_{[m]}$

with $\mathbf{H}_{[m]}[f](x) = |h'_m(x)| \cdot f \circ h_m(x)$

is the density transformer of the dynamical system.

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For a density f on $[0, 1]$,

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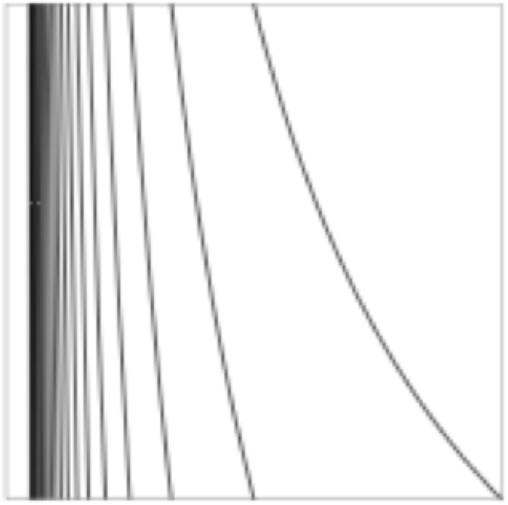

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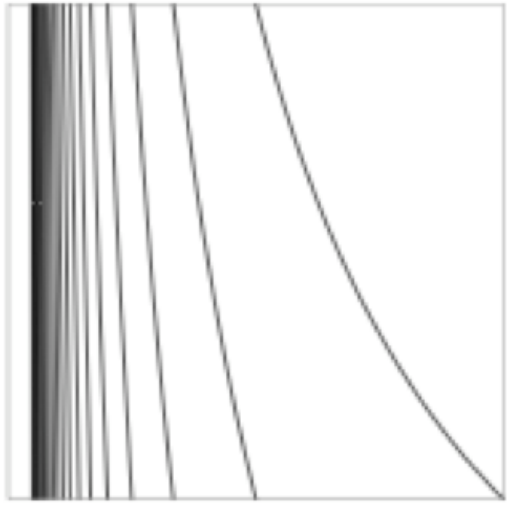

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Transfer operator (Vallée, 2000) [secant version]

$$\mathbb{H}_s := \sum_{m\in\Sigma} \mathbb{H}_{s,[m]} \quad \text{with} \quad \mathbb{H}_{s,[m]}[F](x,y) = \left|\frac{h_m(x) - h_m(y)}{x-y}\right|^s F(h_m(x), h_m(y))$$

Alternative expression of $\Lambda(s)$ in the dynamical case.

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$$\Lambda_k(s) := \sum_{w \in \Sigma^k} p_w^s, \qquad \Lambda(s) := \sum_{w \in \Sigma^\star} p_w^s$$

are "generated" by the secant transfer operator $\mathbb{H}_s$ [V. 2000]

$$\Lambda_k(s) = \mathbb{H}_s^k[L^s](0,1), \quad \Lambda(s) = (I - \mathbb{H}_s)^{-1}[L^s](0,1)$$

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For $s = 1$, $\mathbb{H}_1$ is an extension of $\mathbf{H}$ and has an eigenvalue equal to 1.
For a system of the Good Class, $s \mapsto \Lambda(s)$ has a simple pole at $s = 1$

Part III

– provides sufficient conditions for tameness of dynamical sources

What happens on the left of the vertical line $\Re s = 1$?

It is important for the analysis to deal with a region $\mathcal{R}$ where $\Lambda(s)$ is tame

– it is analytic and of polynomial growth when $\Im s \to \infty$

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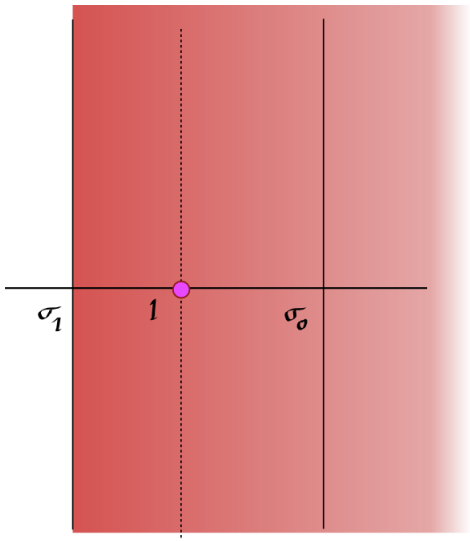
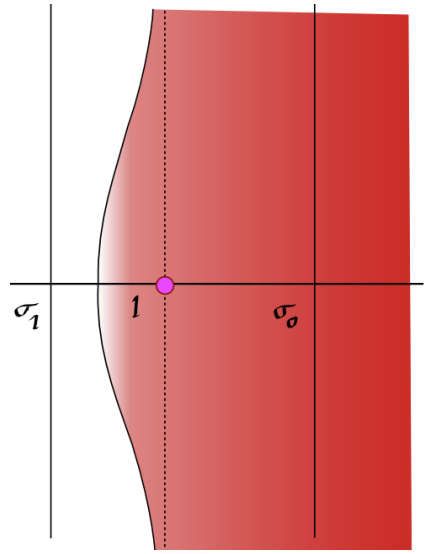


Situation 1

Vertical strip

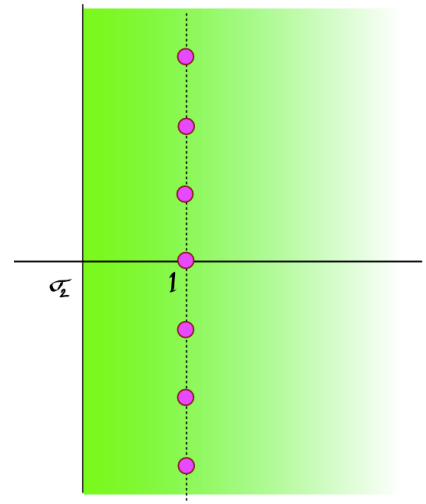
$1 - \sigma \leq a$



Situation 2

Hyperbolic region

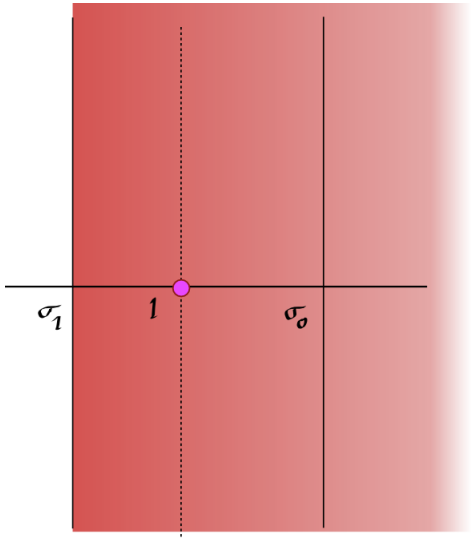
$1 - \sigma \leq t^{-\alpha}$



Situation 3

Vertical strip with holes

Different possible regions on the left of $\Re s = 1$ where $\Lambda(s)$ is tame.



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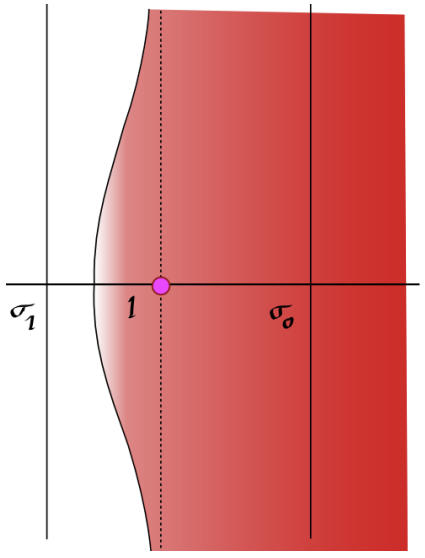

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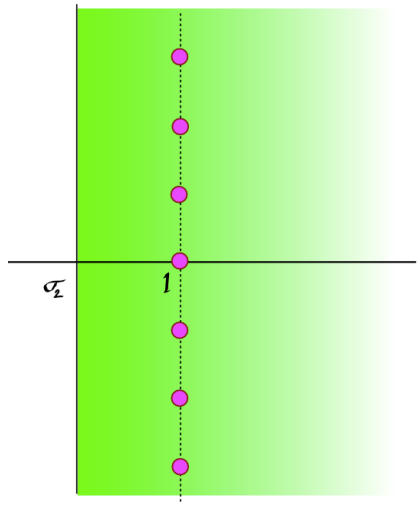

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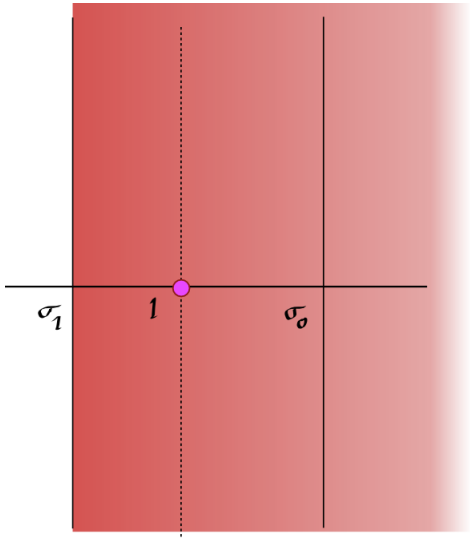

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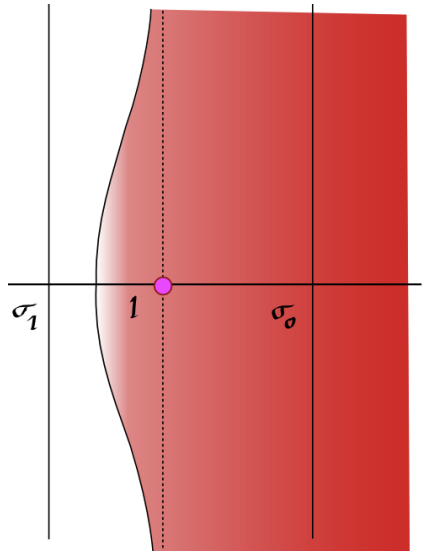
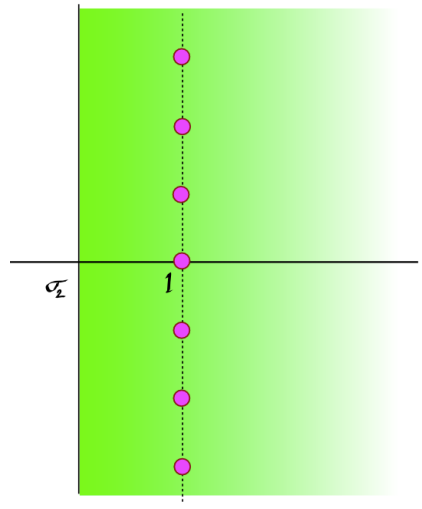


Situation 2

Hyperbolic region



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Vertical strip with holes

For which simple sources do these different situations occur?

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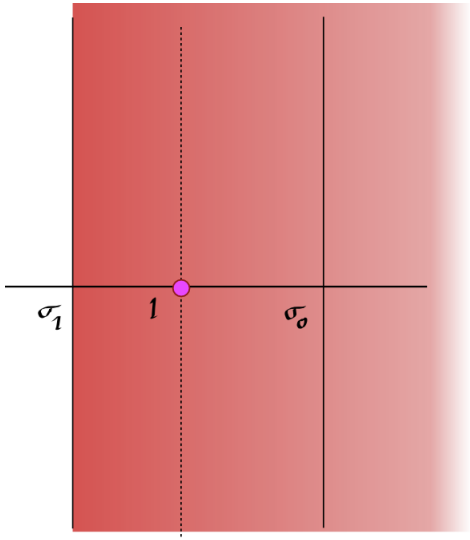

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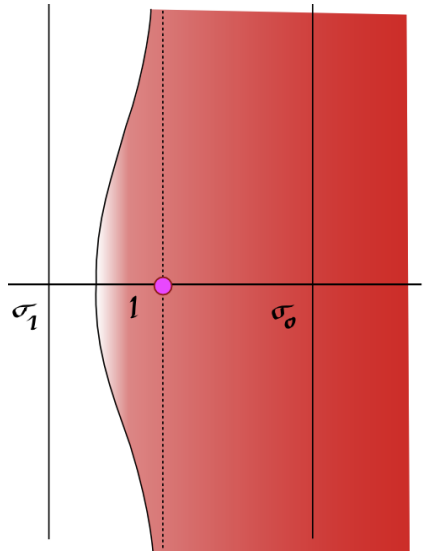

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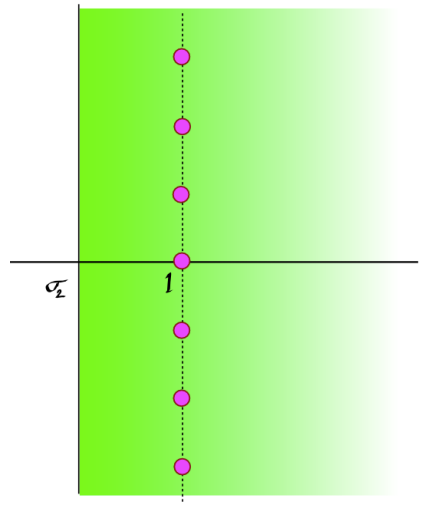

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For which simple sources do these different situations occur?

For memoryless sources relative to probabilities $(p_1, p_2, \ldots, p_r)$

– S1 is impossible

– S3 occurs when all the ratios $\log p_i / \log p_j$ are rational

– S2 occurs if there exists a ratio $\log p_i / \log p_j$
which is "diophantine" [badly approximable by rationals]

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(i) $\mathcal{Z} \neq \{0\}$ $\iff$ $\log p_2/\log p_1$ is rational

(ii) If $t \in \mathcal{Z} \setminus \{0\}$ then $s \mapsto \lambda(s)$ is periodic with period it

(iii) If $\mathcal{Z} = \{0\}$, then the poles of $\Lambda(s)$ close to $\Re s = 1$ are related to good rational approximations of $\log p_2/\log p_1$

Memoryless sources $\Lambda(s) = \dfrac{1}{1-\lambda(s)}$ with $\lambda(s) = p_1^s + p_2^s$ $[r=2]$

The tameness of Λ depends on arithmetical properties of $\log p_2/\log p_1$
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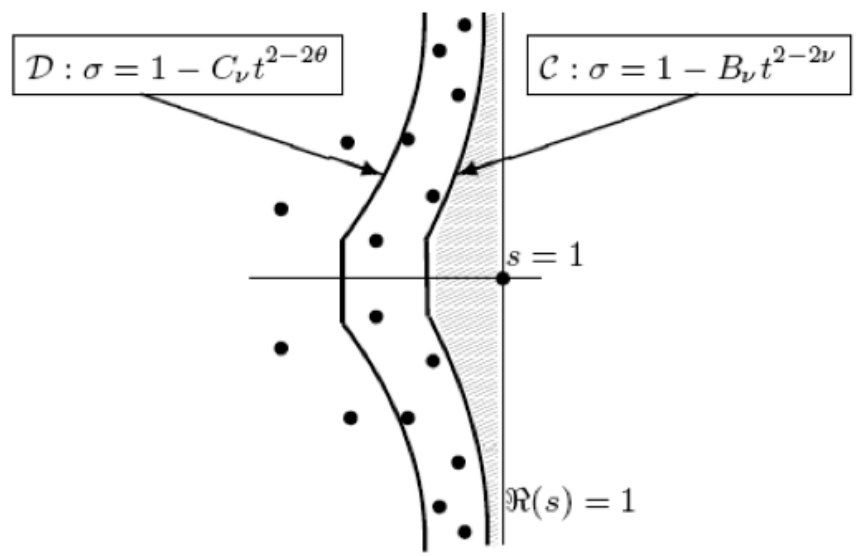
If the irrationality exponent
of $\log p_2/\log p_1$ equals μ
then, for any θ, ν with $\theta < \mu < \nu$,
the tameness region is as shown:
[Flajolet-Roux-V. 2010]



Different possible regions on the left of $\Re s = 1$ where $\Lambda(s)$ is tame.

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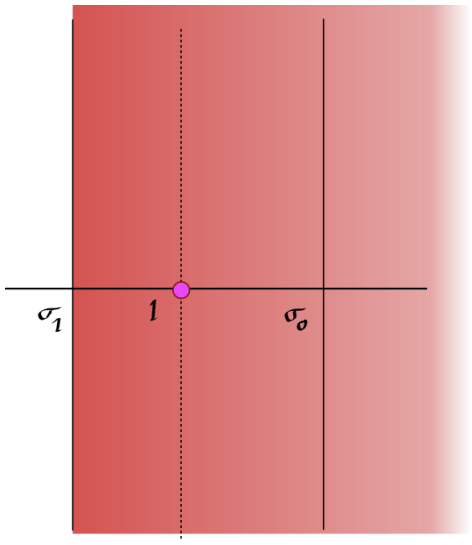
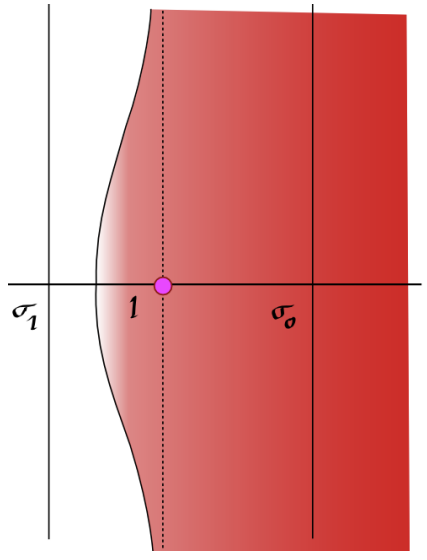


Situation 1

Vertical strip

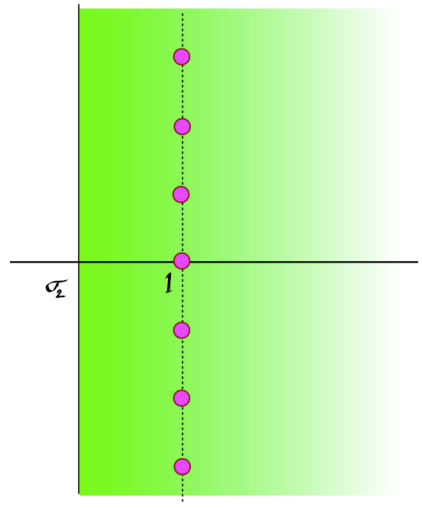
Geometric condition



Situation 2

Hyperbolic region

Arithmetic condition



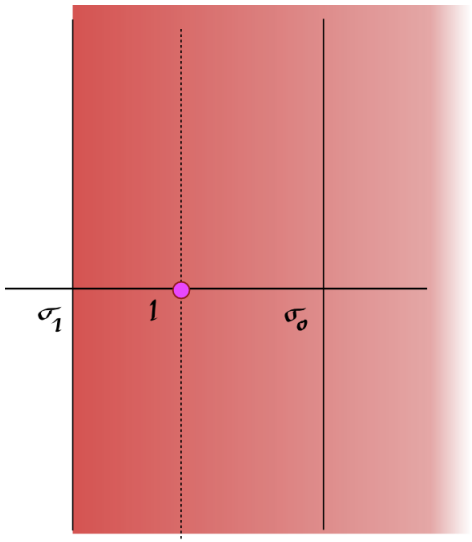
Situation 3

Vertical strip with holes

Periodicity condition

For which general dynamical sources do these different situations occur?

Different possible regions on the left of $\Re s = 1$ where $\Lambda(s)$ is tame.



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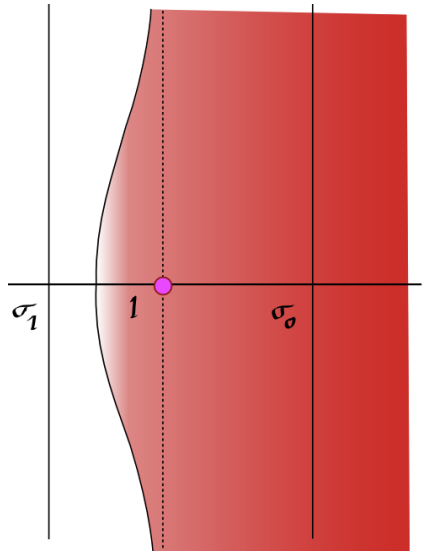
Vertical strip

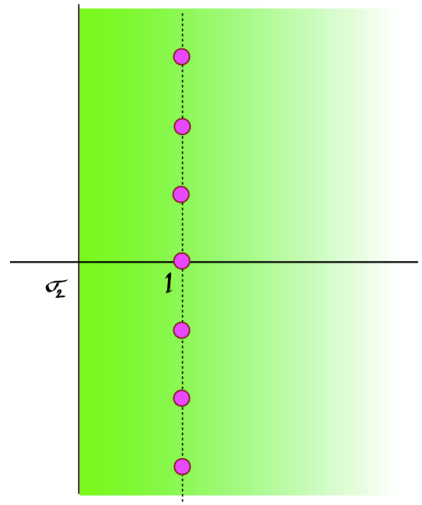
Geometric condition



Situation 2

Hyperbolic region

Arithmetic condition



Situation 3

Vertical strip with holes

Periodicity condition

For which general dynamical sources do these different situations occur?

– S1 occurs when "the branches are not too often of the same shape".

– S3 occurs only if the source is conjugated to a simple source.

– S2 occurs if a extension of the following condition holds:

"there exists a ratio $\log p_i / \log p_j$ which is "diophantine"

Situation 1- Existence of a vertical strip where $\Lambda(s)$ is tame

The condition UNI expresses that

"the branches of the dynamical system are not too often of the same shape"

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For a good dynamical system which satisfies the condition UNI,
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Dolgopyat (98) proves the result for the plain transfer operator, in the case of a finite number of branches

– Baladi and V. (03) extend the result for an infinite number of branches

– Cesaratto and V. (09) extend the result to the secant transfer operator.

Situation 2- Existence of a hyperbolic region where $\Lambda(s)$ is tame

The condition DIOP extends the arithmetic condition

"There exists a ratio $\log p_i / \log p_j$ which is diophantine"

For a complete system, each branch h has a fixed point denoted by $h^\star$.

The derivatives $|h'(h^\star)|$ replace the probabilities of the memoryless case.

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DIOP: There exists a ratio $c(h,k) := \dfrac{\log |h'(h^\star)|}{\log |k'(k^\star)|}$ which is diophantine.

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Dolgopyat (98) proves the result for the plain transfer operator, in the case of a finite number of branches – Roux and V. (2010) extend the result : for an infinite number of branches and for the secant transfer operator.

Conclusion

This talk

– describes a general model for sources

– shows the importance of the Dirichlet generating functions

– explains the importance of tameness in the analyses of text algorithms

– defines a natural subclass of sources, the dynamical sources

– provides sufficient conditions for tameness of dynamical sources

Finally, it provides a precise analysis of text algorithms
when the text is created by a dynamical source.

General result.

When built on a good dynamical source $\mathcal{S}$,

– the mean path-length $T(n)$ of `Trie`,

– the mean symbol path-length $B(n)$ of `Bst`

are both of the form $X(n) = P_X(n) + E(n)$.

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The "principal term" $P_X(n)$ involves the entropy $h(\mathcal{S})$

$$P_T(n) = \frac{1}{h(\mathcal{S})} n \log n + an, \qquad P_B(n) = \frac{1}{h(\mathcal{S})} n \log^2 n + bn \log n + cn,$$

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The order of the "error term" $E(n)$ depends on the tameness of the source

In Situation S1	$E(n) = O(n^{1-\delta})$	$\delta \in [0,1]$
In Situation S2	$E(n) = n \cdot O(\exp[-(\log n)^{\alpha}])$	$\alpha < 1$
In Situation S3	$E(n) = n \cdot \Phi(\log n) + O(n^{1-\delta})$,	$\delta \in [0,1]$ and Φ periodic.