

Enumeration of generalized BCI λ terms

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joint work with
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- 1 λ -terms and enriched trees

- 2 BCI(p)- and BCK(p)-terms
 - BCI(1)- and BCK(1)-terms
 - BCI(p)- and BCK(p)-terms for $p \geq 2$

- 3 General λ -terms

λ -terms and enriched trees

$$T ::= a \mid (T * T) \mid \lambda a.T$$

$(T * T)$: application $\lambda a.T$: abstraction

$(\lambda x.(x * x) * \lambda y.y)$ $\lambda y.(\lambda x.x * \lambda x.y)$

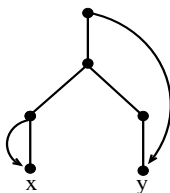
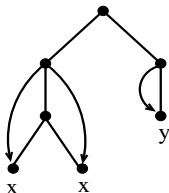
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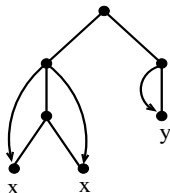


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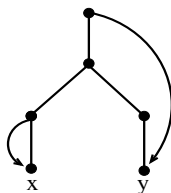
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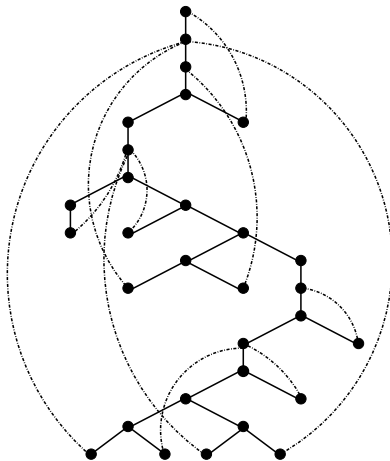
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$\lambda y.(\lambda x.x * \lambda x.y)$



These λ -terms are **closed** (no free variable)

A bigger closed λ -term

- Recursive definition for λ -terms?
 - \mathcal{L} : class of λ -terms with free variables
 - \mathcal{N} atomic class of a binary node
 - \mathcal{U} atomic class of a unary node
 - \mathcal{F} atomic class of a free leaf
 - \mathcal{B} atomic class of a bound leaf

$$\mathcal{L} = \mathcal{F} + \left(\mathcal{N} \times \mathcal{L}^2 \right) + \left(\mathcal{U} \times \text{subs}(\mathcal{F} \rightarrow \mathcal{F} + \mathcal{B}, \mathcal{L}) \right)$$

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■ Generating function

$$L(z, f) = fz + zL(z, f)^2 + zL(z, f + 1).$$

with $z \leftrightarrow$ size of the λ -term and $f \leftrightarrow$ free leaves
(size = total number of nodes)

Generating function enumerating all closed λ -terms: $L(z, 0)$

$$\begin{aligned} L(z, 0) &= [f^0]L(z, f) \\ &= z^2 + 2z^3 + 4z^4 + 13z^5 + 42z^6 + 139z^7 + 506z^8 \\ &\quad + 1915z^9 + 7558z^{10} + \dots \end{aligned}$$

and

$$1 - 2zL(z, 0) =$$

$$\sqrt{1 - 2z + 2z\sqrt{1 - 2z - 4z^2 + 2z\sqrt{\dots\sqrt{1 - 2z - 4nz^2 + 2z\sqrt{\dots}}}}}$$

$L(z, 0)$ has radius of convergence equal to zero

Previous results:

- upper and lower bounds for different notion of size (David, Grygiel, Kozik, Raffalli, Theyssier, Zaionc 2010)
- λ -terms of bounded unary height (Bodini, Gardy, G. 2011)
- BCI and BCK terms (Bodini, Gardy, Jacquot 2010)

BCI(p)- and BCK(p)-terms

Two classes of closed λ -terms:

- $BCI(p)$: each abstraction binds *exactly* p variables
- $BCK(p)$: each abstraction binds *at most* p variables

BCI(1)- and BCK(1)-terms

Generating function for (not necessarily closed) BCI(1)-terms:

$$T(z, f) = zf + zT^2(z, f) + z\frac{\partial T}{\partial f}(z, f)$$

\implies GF for closed terms is $T(z, 0)$

Bodini, Gardy, and Jacquot 2010: Guess and construct bijection: *triangular pointed diagrams*, enumerated according to the number of edges (Vidal 2006)

$$[z^{3n+2}]T(z, 0) \sim \frac{6n(6n/e)^n}{\sqrt{2\pi n}}$$

BCI(p)- and BCK(p)-terms, example $p = 4$

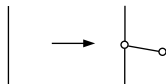
$$\text{BCI}(4) = \text{Diagram 1} + \text{Diagram 2} + \frac{\text{Diagram 3}}{\text{BCI}(4)}$$

The diagram illustrates the recursive definition of BCI(4) terms. It shows three components:

- Diagram 1:** A tree structure with a root node and four children. Each child node has two children of its own, resulting in a total of 16 leaf nodes. This represents a term of the form $\lambda x_1, x_2, x_3, x_4. x_1 x_2 x_3 x_4$.
- Diagram 2:** A simple tree with a root node and two children, representing a term of the form $\lambda x_1, x_2. x_1 x_2$.
- Diagram 3:** A single root node with four children, each of which has a self-loop arrow pointing back to the root. This represents a term of the form $\lambda x_1, x_2, x_3, x_4. x_1 x_2 x_3 x_4$.

BCI(p)- and BCK(p)-terms, example $p = 4$

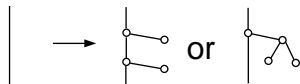
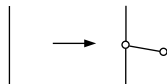
$$\text{BCI}(4) = \begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ \bullet \quad \bullet \quad \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \end{array} + \frac{\begin{array}{c} \bullet \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ \bullet \quad \bullet \quad \bullet \quad \bullet \end{array}}{\text{BCI}(4)}$$



BCI(p)- and BCK(p)-terms, example $p = 4$

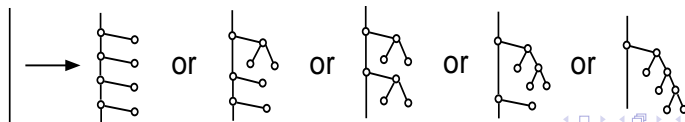
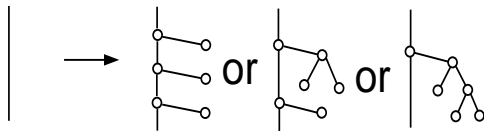
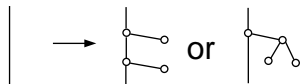
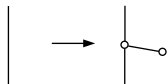
$$\text{BCI}(4) = \text{Diagram 1} + \text{Diagram 2} + \overline{\text{Diagram 3}}$$

The equation defines BCI(4) as the sum of three diagrams. The first diagram is a tree with a root node and four children, each of which has two children of its own. The second diagram is a simple tree with a root node and two children. The third diagram is a root node with four children, each of which has a self-loop arrow pointing back to itself.



BCI(p)- and BCK(p)-terms, example $p = 4$

$$\text{BCI}(4) = \text{BCI}(4) + \text{BCI}(4) + \overline{\text{BCI}(4)}$$



Generating function for $BCI(p)$ defined as solution of a differential equation:

$$Y(z) = \frac{1}{p} \binom{2p-2}{p-1} z^{2p} + zY(z)^2 + \Delta_k Y(z)$$

where Δ_k is the differential operator

$$\Delta_k = \sum_{\ell=1}^k \sum_{\sum_i s_i = \ell; \sum_i i s_i = k} \binom{\ell}{s_1, \dots, s_k} \prod_{m=1}^k \binom{2m}{m}^{s_m} \cdot \frac{z^{\ell+2k+1}}{\ell!} \left(\frac{d}{dz} \right)^\ell$$

GF of $BCK(p)$: First solve

$$\tilde{Y}(z) = \sum_{\ell=1}^k \frac{1}{\ell} \binom{2\ell-2}{\ell-1} z^{2\ell} + z\tilde{Y}(z)^2 + \sum_{\ell=1}^k \Delta_{\ell} \tilde{Y}(z)$$

Then the GF of $BCK(p)$ -terms is $Y(z) = \tilde{Y}\left(\frac{z}{1-z}\right)$

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Lemma

Let $A(z) = \sum_{n \geq n_0} a_n z^n$, $a_{n_0} > 0$ and assume $a_{n+1}/a_n = \Omega(n^p)$ for some $p \geq 2$. Then

$$[z^n]A(z)^2 = 2a_{n_0}a_{n-n_0}(1 + O(n^{-p})), \text{ as } n \rightarrow \infty.$$

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\rightsquigarrow linearization:

$$\tilde{Y}(z) = \frac{1}{p} \binom{2p-2}{p-1} z^{2p} + 2\frac{1}{p} \binom{2p-2}{p-1} z\tilde{Y}(z) + \sum_{\ell=1}^k \Delta_{\ell} \tilde{Y}(z)$$

Theorem

For $p \geq 2$, the number of BCI(p)-terms of size $(2p + 1)n - 1$ is asymptotically

$$C \alpha^n n^\beta (n - 1)!^p$$

where C is a constant and

$$\alpha = \frac{(4p + 2)^p}{p!}, \quad \beta = \frac{p(p - 1)}{4p + 2} \left(17 - \frac{2}{p - 1} \right).$$

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Similar for BCK(p):

$$\# \text{ of terms of size } n \sim \tilde{C} \tilde{\alpha}^{\frac{n}{2p+1}} \left(\frac{n}{2p+1} \right)^\beta (n - 1)!^{\frac{p}{2p+1}}$$

General λ -terms

Theorem

Let $\lambda_n =$ number of λ -terms of size n ,

$$A(z) = \frac{1 - \sqrt{1 - 4z^2}}{2},$$

and $\tilde{\Lambda}(z)$ be the solution of

$$\tilde{\Lambda}(z) = A(z) + z\tilde{\Lambda}\left(\frac{z}{1 - 2A(z)}\right) - z\tilde{\Lambda}(z).$$

Then

$$\Lambda(z) := \sum_{n \geq 2} \lambda_n z^n = \tilde{\Lambda}\left(\frac{z}{1 - z}\right).$$

Proof:

Recall BCK(p):

$$\tilde{Y}(z) = \sum_{\ell=1}^k \frac{1}{\ell} \binom{2\ell-2}{\ell-1} z^{2\ell} + z\tilde{Y}(z)^2 + \sum_{\ell=1}^k \Delta_{\ell} \tilde{Y}(z)$$

and

$$Y(z) = \tilde{Y}\left(\frac{z}{1-z}\right)$$

Thus

$$\tilde{\Lambda}(z) = \sum_{\ell=1}^{\infty} \frac{1}{\ell} \binom{2\ell-2}{\ell-1} z^{2\ell} + z\tilde{\Lambda}(z)^2 + \sum_{\ell=1}^{\infty} \Delta_{\ell} \tilde{\Lambda}(z)$$

and

$$\Lambda(z) = \tilde{\Lambda}\left(\frac{z}{1-z}\right)$$

Proof:

$$\begin{aligned}
 \Delta_k F(z) &= \frac{z^{2k+1}}{k!} \sum_{s_1+\dots+s_k=k} \binom{k}{s_1, \dots, s_k} \prod_{i=1}^k \left(\binom{2i}{i} z \right)^{s_i} \left(\frac{d}{dz} \right)^{s_1+\dots+s_k} F(z) \\
 &= \frac{z^{2k+1}}{k!} \sum_{s_1+\dots+s_k=k} \binom{k}{s_1, \dots, s_k} \left(\left(\frac{d}{dz} \right)^{s_1+\dots+s_k} F \right) (zf(u)) \\
 &\quad \prod_{i=1}^k \left(\frac{1}{i!} \left(\frac{d}{du} \right)^i (zf(u)) \right)^{s_i} \Big|_{u=0} \quad \left(\text{where } f(u) = \frac{1}{\sqrt{1-4u}} \right) \\
 &= \frac{z^{2k+1}}{k!} \left(\frac{d}{du} \right)^k (F(zf(u))) \Big|_{u=0}
 \end{aligned}$$

Asymptotic for λ_n is work in progress

Theorem

For all $\varepsilon > 0$ we have

$$\lambda_n > (n!)^{\frac{1}{2}-\varepsilon}.$$

Furthermore

$$\lambda_n = O\left(\sqrt{n!}\right).$$

Dziękuję!