Enumeration of generalized BCI λ terms

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joint work with
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1. λ-terms and enriched trees
2. BCI(p)- and BCK(p)-terms
   - $BCI(1)$- and $BCK(1)$-terms
   - $BCI(p)$- and $BCK(p)$-terms for $p \geq 2$
3. General λ-terms

λ-terms and enriched trees

$$T ::= a \mid (T * T) \mid \lambda a.T$$

$(T * T)$: application $\lambda a.T$: abstraction

$(\lambda x.(x * x) * \lambda y.y)$ $\lambda y.(\lambda x.x * \lambda x.y)$

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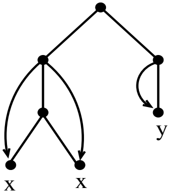

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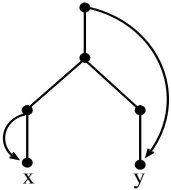

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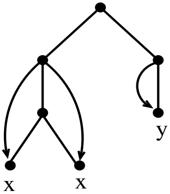

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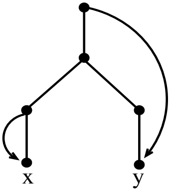
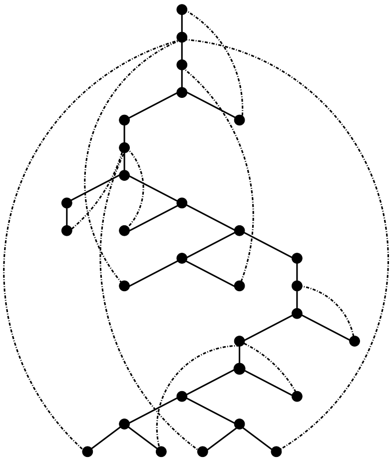


These λ-terms are closed (no free variable)

A bigger closed λ-term



- Recursive definition for λ-terms?
  - $\mathcal{L}$: class of λ-terms with free variables
  - $\mathcal{N}$ atomic class of a binary node
  - $\mathcal{U}$ atomic class of a unary node
  - $\mathcal{F}$ atomic class of a free leaf
  - $\mathcal{B}$ atomic class of a bound leaf

$$\mathcal{L} = \mathcal{F} + \left(\mathcal{N} \times \mathcal{L}^2\right) + (\mathcal{U} \times \mathit{subs}(\mathcal{F} \to \mathcal{F} + \mathcal{B}, \mathcal{L}))$$

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- Generating function

$$L(z, f) = fz + z\,L(z, f)^2 + z\,L(z, f + 1).$$

with $z \leftrightarrow$ size of the λ-term and $f \leftrightarrow$ free leaves
(size = total number of nodes)

Generating function enumerating all closed λ-terms: $L(z,0)$

$$\begin{aligned} L(z,0) &= [f^0]L(z,f) \\ &= z^2 + 2z^3 + 4z^4 + 13z^5 + 42z^6 + 139z^7 + 506z^8 \\ &\quad +1915z^9 + 7558z^{10} + \cdots \end{aligned}$$

and

$$1 - 2zL(z,0) =$$

$$\sqrt{1 - 2z + 2z\sqrt{1 - 2z - 4z^2 + 2z\sqrt{\ldots.\sqrt{1 - 2z - 4nz^2 + 2z\sqrt{\ldots}}}}}$$

$L(z,0)$ has radius of convergence equal to zero

Previous results:

- upper and lower bounds for different notion of size (David, Grygiel, Kozik, Raffalli, Theyssier, Zaionc 2010)
- λ-terms of bounded unary height (Bodini, Gardy, G. 2011)
- BCI and BCK terms (Bodini, Gardy, Jacquot 2010)

BCI(p)- and BCK(p)-terms

Two classes of closed λ-terms:

- $BCI(p)$: each abstraction binds *exactly* p variables
- $BCK(p)$: each abstraction binds *at most* p variables

BCI(1)- and *BCK*(1)-terms

Generating function for (not necessarily closed) BCI(1)-terms:

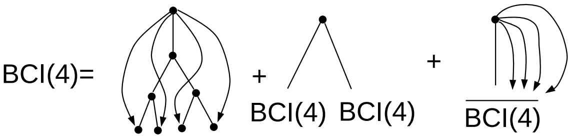
$$T(z,f) = zf + zT^2(z,f) + z\frac{\partial T}{\partial f}(z,f)$$

$\Longrightarrow$ GF for closed terms is $T(z,0)$

Bodini, Gardy, and Jacquot 2010: Guess and construct bijection: *triangular pointed diagrams*, enumerated according to the number of edges (Vidal 2006)

$$[z^{3n+2}]T(z,0) \sim \frac{6n(6n/e)^n}{\sqrt{2\pi n}}$$

BCI(p)- and *BCK*(p)-terms, example $p = 4$



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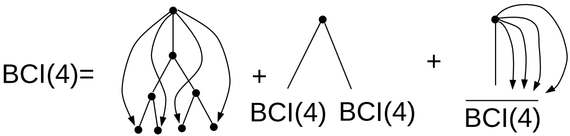

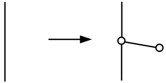

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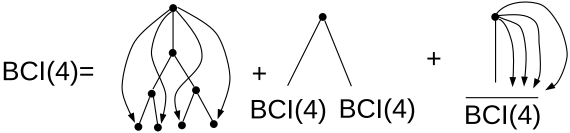
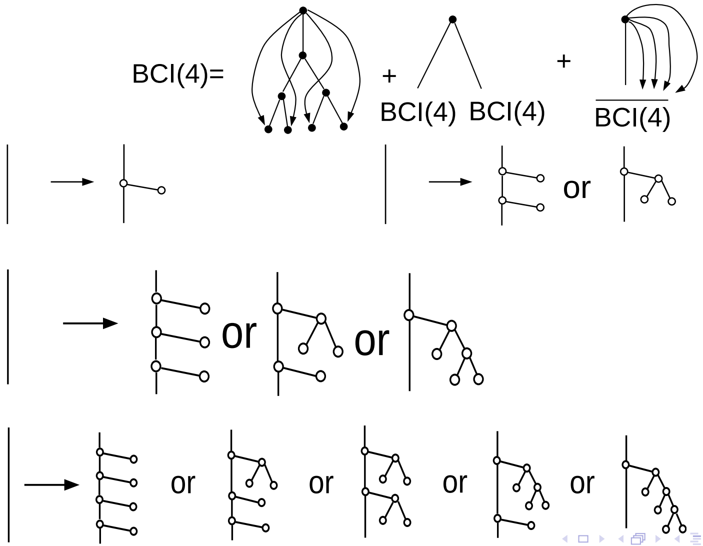


BCI(*p*)- and *BCK*(*p*)-terms, example $p = 4$



Generating function for *BCI*(p) defined as solution of a differential equation:

$$Y(z) = \frac{1}{p}\binom{2p-2}{p-1} z^{2p} + zY(z)^2 + \Delta_k Y(z)$$

where Δ_k is the differential operator

$$\Delta_k = \sum_{\ell=1}^{k} \sum_{\sum_i s_i = \ell; \sum_i i s_i = k} \binom{\ell}{s_1, \ldots, s_k} \prod_{m=1}^{k} \binom{2m}{m}^{s_m} \cdot \frac{z^{\ell+2k+1}}{\ell!} \left(\frac{\mathrm{d}}{\mathrm{dz}}\right)^{\ell}$$

GF of $BCK(p)$: First solve

$$\tilde{Y}(z) = \sum_{\ell=1}^{k} \frac{1}{\ell}\binom{2\ell-2}{\ell-1} z^{2\ell} + z\tilde{Y}(z)^2 + \sum_{\ell=1}^{k} \Delta_\ell \tilde{Y}(z)$$

Then the GF of $BCK(p)$-terms is $Y(z) = \tilde{Y}\left(\frac{z}{1-z}\right)$

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Lemma

Let $A(z) = \sum_{n \geq n_0} a_n z^n$, $a_{n_0} > 0$ *and assume* $a_{n+1}/a_n = \Omega(n^p)$ *for some* $p \geq 2$. *Then*

$$[z^n]A(z)^2 = 2a_{n_0} a_{n-n_0}(1 + O(n^{-p})), \text{ as } n \to \infty.$$

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$\rightsquigarrow$ linearization:

$$\tilde{Y}(z) = \frac{1}{p}\binom{2p-2}{p-1} z^{2p} + 2\frac{1}{p}\binom{2p-2}{p-1} z\tilde{Y}(z) + \sum_{\ell=1}^{k} \Delta_\ell \tilde{Y}(z)$$

Theorem

For $p \geq 2$, the number of BCI(p)-terms of size $(2p+1)n-1$ is asymptotically

$$C\,\alpha^n n^\beta (n-1)!^p$$

where C is a constant and

$$\alpha = \frac{(4p+2)^p}{p!}, \qquad \beta = \frac{p(p-1)}{4p+2}\left(17 - \frac{2}{p-1}\right).$$

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Similar for BCK(p):

$$\#\text{ of terms of size } n \sim \tilde{C}\,\tilde{\alpha}^{\frac{n}{2p+1}} \left(\frac{n}{2p+1}\right)^\beta (n-1)!^{\frac{p}{2p+1}}$$

General λ-terms

Theorem

Let $\lambda_n =$ *number of* λ*-terms of size* n*,*

$$A(z) = \frac{1 - \sqrt{1 - 4z^2}}{2},$$

and $\tilde{\Lambda}(z)$ *be the solution of*

$$\tilde{\Lambda}(z) = A(z) + z\tilde{\Lambda}\left(\frac{z}{1 - 2A(z)}\right) - z\tilde{\Lambda}(z).$$

Then

$$\Lambda(z) := \sum_{n \geq 2} \lambda_n z^n = \tilde{\Lambda}\left(\frac{z}{1 - z}\right).$$

Proof:
Recall BCK(p):

$$\tilde{Y}(z) = \sum_{\ell=1}^{k} \frac{1}{\ell}\binom{2\ell-2}{\ell-1} z^{2\ell} + z\tilde{Y}(z)^2 + \sum_{\ell=1}^{k} \Delta_\ell \tilde{Y}(z)$$

and

$$Y(z) = \tilde{Y}\left(\frac{z}{1-z}\right)$$

Thus

$$\tilde{\Lambda}(z) = \sum_{\ell=1}^{\infty} \frac{1}{\ell}\binom{2\ell-2}{\ell-1} z^{2\ell} + z\tilde{\Lambda}(z)^2 + \sum_{\ell=1}^{\infty} \Delta_\ell \tilde{\Lambda}(z)$$

and

$$\Lambda(z) = \tilde{\Lambda}\left(\frac{z}{1-z}\right)$$

Proof:

$$\Delta_k F(z)$$

$$= \frac{z^{2k+1}}{k!} \sum_{s1+\ldots+ks_k=k} \binom{k}{s_1,\ldots,s_k} \prod_{i=1}^{k} \left(\binom{2i}{i} z \right)^{s_i} \left(\frac{\mathrm{d}}{\mathrm{dz}} \right)^{s_1+\ldots+s_k} F(z)$$

$$= \frac{z^{2k+1}}{k!} \sum_{s1+\ldots+ks_k=k} \binom{k}{s_1,\ldots,s_k} \left(\left(\frac{\mathrm{d}}{\mathrm{dz}} \right)^{s_1+\ldots+s_k} F \right) (zf(u))$$

$$\prod_{i=1}^{k} \left(\frac{1}{i!} \left(\frac{\mathrm{d}}{\mathrm{du}} \right)^{i} (zf(u)) \right)^{s_i} \Bigg|_{u=0} \quad \left(\text{where } f(u) = \frac{1}{\sqrt{1-4u}} \right)$$

$$= \frac{z^{2k+1}}{k!} \left(\frac{\mathrm{d}}{\mathrm{du}} \right)^{k} (F(zf(u))) \Bigg|_{u=0}$$

Asymptotic for λ_n is work in progress

Theorem

For all $\varepsilon > 0$ *we have*

$$\lambda_n > (n!)^{\frac{1}{2}-\varepsilon}.$$

Furthermore

$$\lambda_n = O\left(\sqrt{n!}\right).$$

Dziękuję!