Associative and commutative tree representation for Boolean functions

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1. Tree representation for Boolean functions
   - Plane binary model
   - Commutative or associative models

2. Probability distributions on Boolean functions
   - Relation between complexity and probability
   - Key ideas of the proof

3. Conclusion and perspectives

Tree representation for Boolean functions

Boolean formulas

- A set of literals: $\{x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n\}$
- A set of connectors: $\{\wedge, \vee\}$

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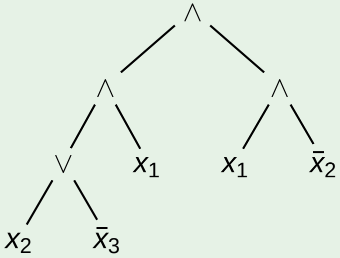

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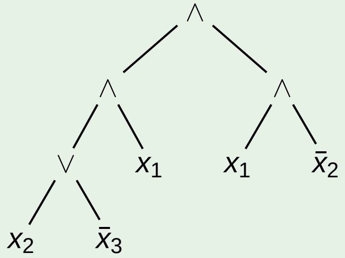

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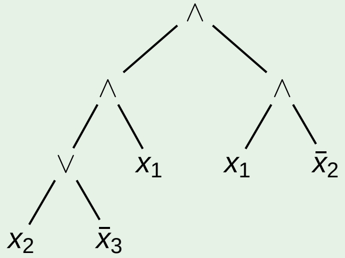

- The set of all plane binary and labelled trees: $\mathcal{T}_n$
- The **size** of a tree corresponds to its number of leaves

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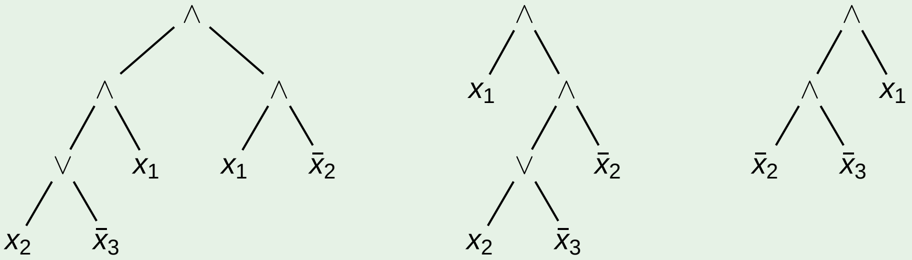

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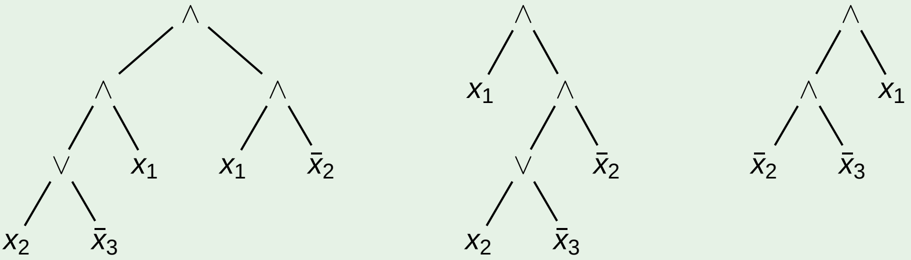

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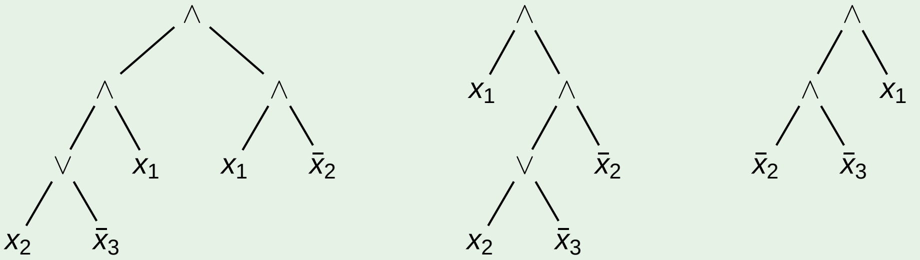

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- The **complexity** of a function is the size of its smallest trees

Probability distribution

- $\mathcal{A}$ a subset of $\mathcal{T}_n$
- $\mathcal{A}(m)$ the set of all formulas of size m in $\mathcal{A}$

[Lefman, Savický. *Some typical properties of large And / Or boolean formulas*, 1997]

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- Using Drmota-Lalley-Woods Theorem:

$\mathbb{P}_n(\cdot)$ is a probability distribution on $\mathcal{F}_n$.

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Previous results

Let $f \in \mathcal{F}_n$ be a function of complexity $L(f)$.

And/Or plane binary trees

$$\mathbb{P}_n(f) = \Theta\left(\frac{1}{n^{L(f)+1}}\right), \text{ when } n \text{ tends to infinity.}$$

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Remark: Formulas with symmetries have a smaller limiting ratio than others.

Natural properties of the connectors

Let us take into account some properties of the operators.
What about:

associativity or

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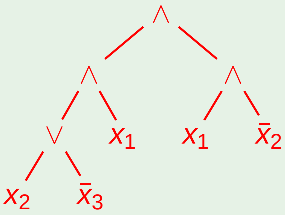
- How does the tree structure evolve with such properties?
- Which behaviour for the new distributions on Boolean functions?

Tree models

connectors without associativity and commutativity	plane binary trees



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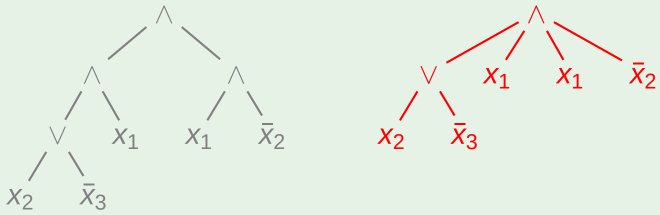

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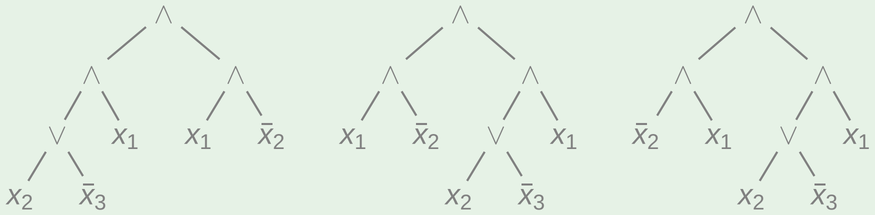

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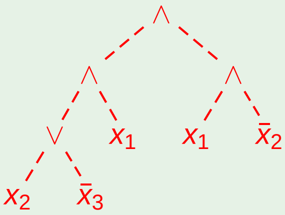

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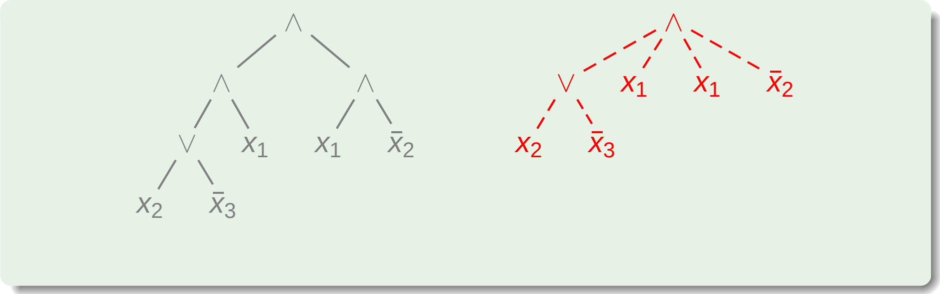

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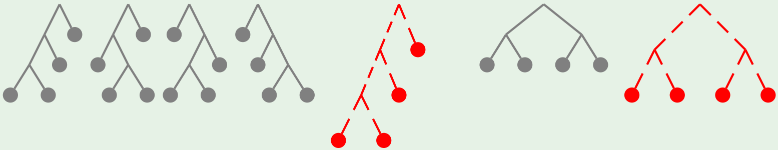
Remark: The commutative models increase the limiting ratio of symmetric formulas.



Probability distributions on Boolean functions

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Theorem

Let us fix one of the four tree models, then

$$\mathbb{P}_n(f) = \frac{\lambda_f}{n^{L(f)+1}} + \mathcal{O}\left(\frac{1}{n^{L(f)+2}}\right), \text{ when } n \text{ tends to infinity.}$$

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Corollary: Most trees computing f have a simple structure.

Key ideas [Kozik; Fournier *et al*]

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1. Definition of **simple family** of trees for f
2. All other trees for f:
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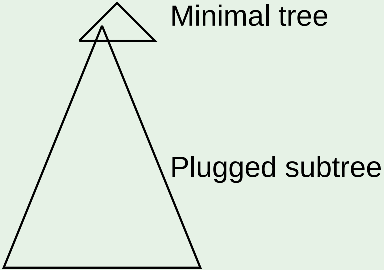
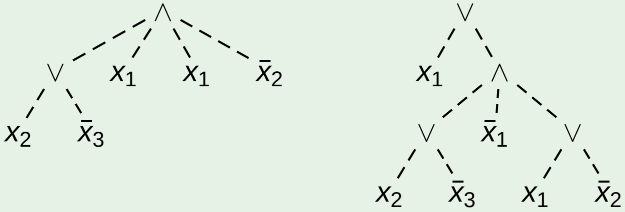
3. **Theorem:** Almost all trees computing the function f are obtained by plugging a special subtree in a minimal tree of f.



The associative and commutative model



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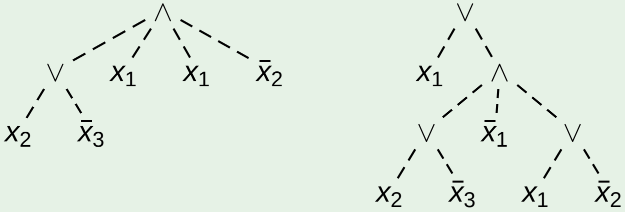

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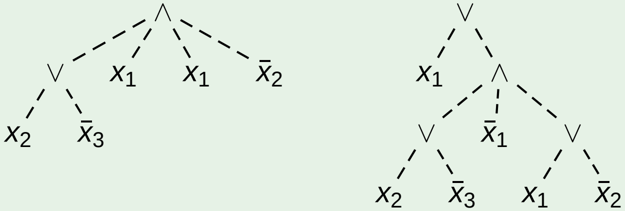

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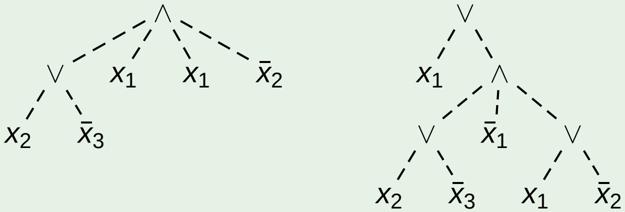

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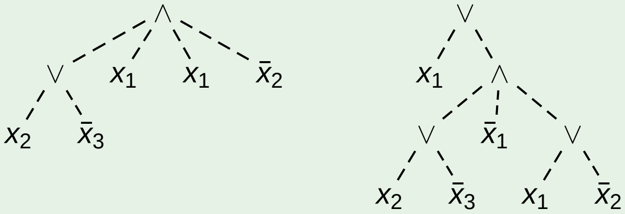

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Main lemma

Let $\mathcal{V}$ be a fixed subset of literals and
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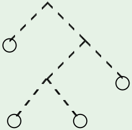

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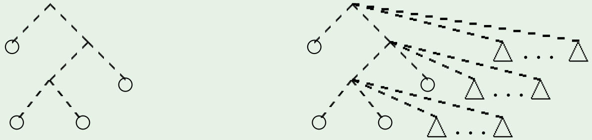

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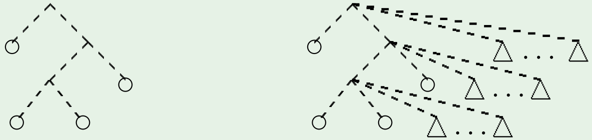

Lemma

$$\mu_n(\mathcal{F}_A) = \Theta\left(\frac{1}{n^{|A|}}\right),$$ when n tends to infinity.

The proof is adapted from Kozik's proof on plane binary trees.

Valid expansions

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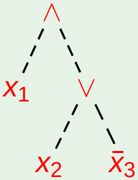

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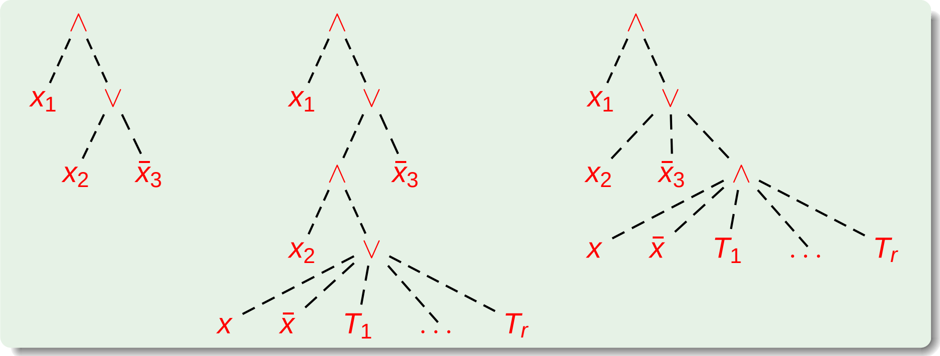

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All other trees computing f belong to families containing
at least 5 restrictions.

Conclusion and perspectives

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Conclusion

- Without closed forms for the generating functions of the families of trees we consider, we have obtained the same kind of results than with plane binary trees.
- Although the commutative models give more importance to symmetric formulas, it is not sufficient for modifying the asymptotic order of the probability of fixed functions.
- According to these results on probabilities on fixed functions, we conjecture to have the same behaviour:
  Almost all functions are of low complexity.

Perspectives

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- Using direct acyclic graphs instead of trees.