Two-sided tail bounds for thin tailed perpetuities

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Iterating the above equation yields

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Assuming that the first term is negligible and...

...re–numbering (Q_n, M_n) in the opposite direction we see that

$$R_n \overset{d}{\sim} \sum_{i=1}^{n} Q_i \prod_{j=1}^{i-1} M_j,$$

so that R may be defined as

$$R \overset{d}{=} \sum_{i=1}^{\infty} Q_i \prod_{j=1}^{i-1} M_j,$$

provided that the series converges at least in distribution.

Kesten (1973) showed that

$$E \log^{+} |Q| < \infty \quad \text{and} \quad E \log |M| < 0$$

suffice for the almost sure convergence of the series

$$\sum_{j=1}^{\infty} Q_j \prod_{k=1}^{j-1} M_k .$$

And for the almost sure convergence to 0 of

$$R_0 \prod_{j=1}^{n} M_j .$$

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If

$$E|M|^\kappa \log^+ |M| < \infty, \quad E|Q|^\kappa < \infty,$$

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This basic result has been re–proved and extended by a number of researchers, among others Goldie (1991), Grey (1994), Grincievičjus (1975)

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For example:

- in the context of record times of random random walks Vervaat (1972) studied the situation in which

$$Q \equiv 1 \quad \text{and} \quad M \stackrel{d}{=} \mathrm{beta}(\alpha, 1) \stackrel{d}{=} U^{1/\beta},$$

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- AofA regulars who wrote on perpetuities include (in order of appearance at AofA'11): Fill, Grübel, Mahmoud, Neininger, Hwang, Rösler; Devroye.

Light tails, cont.

- in the special case $\alpha = 1$ Vervaat perpetuity is distributed like a Dickman function $\rho(u)$ appearing in number theory, namely

$$\rho(u) = \lim_{n\to\infty} \frac{k_n(u)}{n}$$

where $k_n(u)$ is the number of positive integers $\leq n$ with the largest prime factor no more than $n^{1/u}$.

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- other appearances of Dickman function are discussed in Hwang and Tsai (2001) and include the analysis of `Quickselect` algorithm, the degree of the largest irreducible factor in a random polynomial over finite field, and allele frequencies in some biological models.

Light tails, cont.

However, much less precise information is available in this case.

- For Vervaat perpetuities R Vervaat showed that

$$\frac{\log P(R > x)}{x \log x} \sim -1, \quad \text{as} \quad x \to \infty.$$

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- Goldie and Grübel (1996) were the first to study light–tailed case in some generality. They showed that:
- the tails are no heavier than exponential
- if $Q \equiv q > 0$ and $0 \leq M \leq 1$ satisfies:

$$c\delta \leq P(1 - \delta \leq M \leq 1) \leq C\delta,$$

for some $\epsilon > 0$, $0 < c, C < \infty$ and for all $\delta \in (0, \epsilon]$ then

$$\lim_{x \to \infty} \frac{\ln(P(R \geq x))}{x \ln x} = -\frac{1}{q}.$$

Light tails, cont.

- H. and Wesołowski (2009) extended these ideas to construct M's for which the corresponding R satisfies, for example:

$$\frac{\ln P(R > x)}{x \ln x} \sim -\frac{\beta}{q}, \quad \text{as} \quad x \to \infty, \quad \beta > 0.$$

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- From a different perspective Jurek (1999) showed that *every* c-decomposable random variable X can be written as a perpetuity:

$$X \stackrel{d}{=} e^{-\tau} X + X_\tau,$$

where $\tau \geq 0$ and (τ, X_τ) is independent of X on the rhs.

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$$X \stackrel{d}{=} e^{-\tau} X + X_\tau,$$

where $\tau \geq 0$ and (τ, X_τ) is independent of X on the rhs. X is c-decomposable if $\forall c \in [0,1]\ \exists X_c :\ X \stackrel{d}{=} cX + X_c$, with X and X_c independent on the rhs.

Light tails: general lower bound

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Then, when $Q \equiv q > 0$, for $y > 0$ we have

$$P(R \ge \frac{q}{\delta}(1-(1-\delta)^y)) \ge p_\delta^y.$$

In particular, if for $c \in (0,1)$ and $x > q$ we let

$$\delta = \frac{cq}{x} \quad \text{and} \quad y = \frac{\ln(1-c)}{\ln(1-cq/x)},$$

then

$$\begin{aligned} P(R \ge x) &\ge \exp\left(\frac{\ln(1-c)}{\ln(1-cq/x)}\ln(p_{\frac{cq}{x}})\right) \\ &\sim \exp\left(-\frac{\ln(1-c)}{c}\frac{x}{q}\ln(p_{\frac{cq}{x}})\right). \end{aligned}$$

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Theorem: *There exist constants* c_1, c_2 *such that if* $|Q| \leq q$ *and* $|M| \leq 1$ *then for sufficiently large* x*:*

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Theorem: *There exist constants* c_1, c_2 *such that if* $|Q| \leq q$ *and* $|M| \leq 1$ *then for sufficiently large* x*:*

$$P(|R| > x) \leq \exp(\frac{c_1}{q} x \ln p_{c_2 q/x}).$$

In particular, if $Q \equiv q > 0$ *and* $0 \leq M \leq 1$ *then*

$$\exp(\frac{2 \ln 2}{q} x \ln p_{q/(2x)}) \leq P(R > x) \leq \exp(\frac{c_1}{q} x \ln p_{c_2 q/x}).$$

Comments on proof

- techniques for the cases $0 \leq M \leq 1$ and $P(M > 1) > 0$ are completely different.
- techniques previously used for an upper bound in the case $0 \leq M \leq 1$ were generally based on an iteration of the equation $R_n \stackrel{d}{=} M_n R_{n-1} + Q_n$ and they don't seem to work.
- However, a proof of a lower bound of Goldie–Grübel may be used to yield an upper bound.

Goldie-Grübel lower bound

For $0 < \delta < 1$ let

$$T = \inf\{k \geq 1 : \; M_k \leq 1 - \delta\}.$$

Then $T > n$ implies that

$$\begin{aligned} R &\geq q(1 + (1-\delta) + (1-\delta)^2 + \cdots + (1-\delta)^n) \\ &= \frac{q}{\delta}(1 - (1-\delta)^{n+1}), \end{aligned}$$

Hence

$$P(R \geq \frac{q}{\delta}(1 - (1-\delta)^{n+1})) \geq p_\delta^n,$$

and hence by considering $n \leq y \leq n+1$ for $y > 0$

$$P(R \geq \frac{q}{\delta}(1 - (1-\delta)^y)) \geq p_\delta^y.$$

Setting $\delta = \frac{cq}{x}$ and $y = \frac{\ln(1-c)}{\ln(1-cq/x)}$ yields

$$P(R \geq x) \geq \exp\left(\frac{\ln(1-c)}{\ln(1-cq/x)} \ln(p_{\frac{cq}{x}})\right) \sim \exp\left(-\frac{\ln(1-c)}{c}\frac{x}{q}\ln(p_{\frac{cq}{x}})\right).$$

General upper bound

Assume wlog that $P(M=1)=0$ for otherwise, for sufficiently small δ, $p_\delta = P(M=1)$ is a positive constant and the bound is comparable to a tail of an exponential random variable.
For a fixed $0<\delta<1$ define a sequence of stopping times (T_k) by:

$$T_0 = 0, \quad T_m = \inf\{k \geq 1 : \ |M_{T_{m-1}+k}| \leq 1-\delta\}, \quad m \geq 1.$$

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Then, T_k's are i.i.d. geometric with parameter $1-p_\delta$.
Furthermore, $|M_k| \leq 1-\delta$ if $k = T_1 + \cdots + T_i$ for some $i \geq 1$ and $|M_k| \leq 1$ otherwise.

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Furthermore, $|M_k| \leq 1 - \delta$ if $k = T_1 + \cdots + T_i$ for some $i \geq 1$ and $|M_k| \leq 1$ otherwise. Hence,

$$\prod_{k=1}^{m} |M_k| \leq (1-\delta)^j \quad \text{when} \quad T_1 + \cdots + T_j \leq m < T_1 + \cdots + T_j + T_{j+1}.$$

Proof of upper bound - 2

Consequently,

$$\begin{aligned}\left|\sum_{k\geq 1}\prod_{j=1}^{k-1} M_k\right| &\leq \sum_{k\geq 1}\prod_{j=1}^{k-1} |M_k| \leq T_1 + (1-\delta)T_2 + (1-\delta)^2 T_3 + \cdots \\ &= \sum_{k\geq 1}(1-\delta)^{k-1} T_k.\end{aligned}$$

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Hence, $|Q| \leq q$ implies

$$P(|R| > x) \leq P(\sum_{k\geq 1}\prod_{j=1}^{k-1} |M_j| \geq \frac{x}{q}) \leq P(\sum_{k\geq 1} T_k(1-\delta)^{k-1} \geq \frac{x}{q}).$$

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At this point we have a sum of independent random variables and we follow a standard path.

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$$P(\sum_{k\geq 1}(1-\delta)^{k-1}T_k \geq t) \leq e^{-\lambda t} Ee^{\lambda \sum_{k\geq 1} T_k (1-\delta)^{k-1}}$$

$$\leq \exp(-\lambda t + \frac{\lambda}{\delta} + 2\frac{p}{1-p}\sum_{k\geq 1}(e^{\lambda(1-\delta)^{k-1}} - 1)).$$

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THANKS :) :)