A generalization of the Erdős-Turán law for the order of random permutation

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joint work with Alexander Gnedin (Utrecht, Netherlands) and Alexander Marynych (Kiev, Ukraine)

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Overview

- basic notions
    - cycles of permutations
    - order of permutation
    - Ewens' permutations
    - classical Erdős-Turán law

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- permutations derived from stick-breaking
  - basic construction
  - a Markov chain representation
  - the infinite occupancy scheme

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- permutations derived from stick-breaking
  - basic construction
  - a Markov chain representation
  - the infinite occupancy scheme
- extension of the Erdős-Turán law
  - main result
  - scheme of the proof
  - weak convergence of the product of cycle lengths
  - asymptotics of the cycle partition

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$$\pi = (1\,9\,6\,2)(3\,7\,5)(4\,8) \quad \longrightarrow \quad O(\pi) = 12$$

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$$\mathbb{P}\{\Pi_n = \pi\} = \theta^{|\pi|}/(\theta)_n, \quad \theta > 0, \ \pi \in S_n,$$

where $|\pi|$ denotes the number of cycles of π, and $(\theta)_n := \Gamma(\theta + n)/\Gamma(\theta)$.

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The Ewens sampling formula:

$$\mathbb{P}\{(K_{n,1}, \ldots, K_{n,n}) = (k_1, \ldots, k_n)\}$$
$$= \quad 1_{\{\sum_{j=1}^n jk_j = n\}} \frac{n!}{\theta(n)} \prod_{j=1}^{n} \left(\frac{\theta}{j}\right)^{k_j} \frac{1}{k_j!}, \quad k_j = 0, 1, \ldots, n,$$

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where $K_{n,r}$ denotes the number of cycles of length r.

The instance $\theta = 1$ corresponds to the uniform random permutation (arises in the Hat-Check-Problem).

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Classical Erdős- Turán law

Let O_n denote the order of random permutation Π_n, i.e., the least common multiple of the different cycle lengths of Π_n.

- Erdős & Turán (1967): for the uniform random permutation

$$\frac{\log O_n - (1/2)\log^2 n}{\sqrt{(1/3)\log^3 n}} \overset{d}{\to} \mathcal{N}(0,1), \qquad n \to \infty.$$

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  The proof uses Feller coupling and asymptotic independence of cycle lengths.

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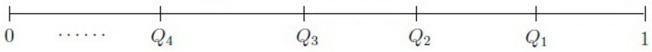

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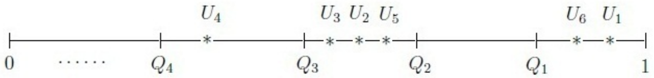

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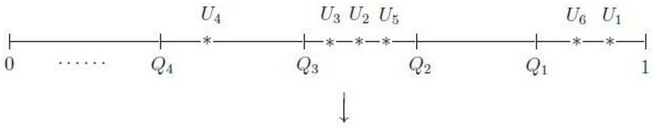

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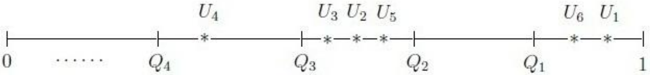

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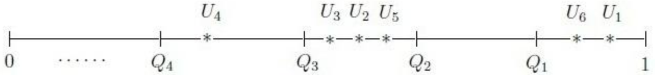

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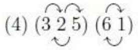

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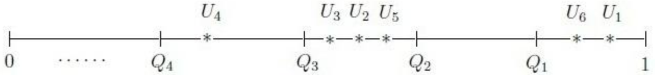

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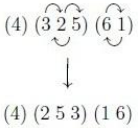

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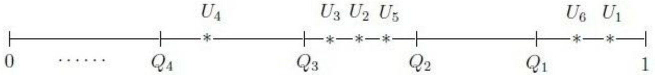

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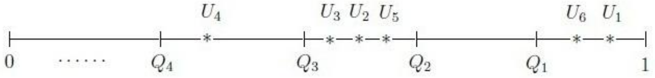

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- Regeneration : For $m \in \{1, \ldots, n-1\}$, conditionally given the last cycle of Π_n has length m, the cycle partition of Π_n with the last cycle deleted has the same distribution as the cycle partition of Π_{n-m}.

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Regeneration+exchangeability $\rightarrow$ given the last cycle of Π_n is of length m, a permutation resulting from deletion of the last cycle and re-labelling the remaining elements by the increasing bijection with $[n-m]$ is a distributional copy of Π_{n-m}.

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For the permutations derived from stick-breaking:

- S is a compound Poisson process with jumps $(|\log W_k|)$;
- The length of the last cycle of Π_n is of order $\Theta(n)$, as n grows.

Permutations derived from steak-breaking:
A Markov chain representation

$K_{n,r}$ is the number of cycles of length r.

Consider a decreasing Markov chain on nonnegative integers with absorption at 0 and the decrement matrix

$$q(n,m) = \binom{n}{m} \frac{\mathbb{E}[W^{n-m}(1-W)^m]}{1-\mathbb{E}W^n}, \quad 1 \le m \le n,$$

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For the Markov chain starting at n, $K_{n,r}$ is the number of decrements of size r on the path from n to 0.

Permutations derived from stick-breaking:
The infinite occupancy scheme

$$Q_0 := 1, \quad Q_j := \prod_{i=1}^{j} W_i, \quad j \in \mathbb{N},$$

where (W_i) are independent copies of the stick-breaking factor W.

For $j \in \mathbb{N}$, the interval $(Q_j, Q_{j-1}]$ = box with index j and frequency

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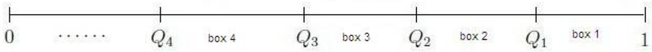

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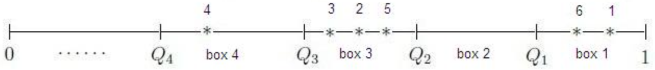

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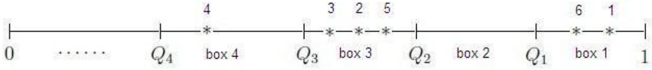
Then $K_{n,r}$ is the number of boxes occupied by exactly r out of the first n balls.



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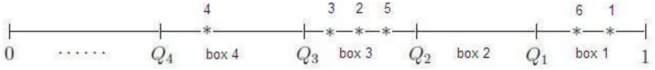
$$K_{6,4} = 0, \quad K_{6,3} = 1$$



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The model was investigated by

- Gnedin (2004),
- Gnedin, I. & Roesler (2008),
- Gnedin, I., Roesler & Negadajlov (2009),
- Gnedin, I. & Marynych (2010a,b),
- I. (2011+).

Extension of the Erdős-Turán law:
Main result

Let O_n denote the order of random permutation Π_n derived from stick-breaking of $[0,1]$, i.e.,

$$O_n = \text{l.c.m.}\{r \in [n] : K_{n,r} > 0\},$$

where $K_{n,r}$ is the number of cycles of Π_n with length r.

Theorem

Suppose the stick-breaking factor W has density f which satisfies

$$\sup_{x\in[0,1]} x^\alpha (1-x)^\alpha f(x) < \infty,$$

for some $\alpha \in [0,1)$. Then $\sigma^2 := \text{Var}\,(\log W) < \infty$ and

$$\frac{\log O_n - (2\mu)^{-1}\log^2 n}{\sqrt{(3\mu^3)^{-1}\sigma^2 \log^3 n}} \overset{d}{\to} \mathcal{N}(0,1), \quad n \to \infty,$$

where $\mu := \mathbb{E}|\log W|$.

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Applicability : Theorem applies if, for instance,

- W has a bounded density;
- W has the beta (a,b) density, $a,b>0$:

$$f(x) = \frac{1}{\mathrm{B}(a,b)} x^{a-1}(1-x)^{b-1} 1_{(0,1)}(x).$$

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! If $W \stackrel{d}{=} \mathrm{beta}(\theta, 1)$, $\theta > 0$ we recover the Erdős-Turán law for Ewens' permutations.

Extension of the Erdős-Turán law:
Scheme of the proof

$$O_n = \text{l.c.m.}\{r \in [n] : K_{n,r} > 0\} = \text{l.c.m.}\{Z_{n,k} : k \le K_n\},$$

where $Z_{n,k}$ is the length of the kth cycle of Π_n, and $K_n := \sum_{r=1}^{n} K_{n,r}$ is the total number of cycles of Π_n.

STEP 1: Show that $\log T_n$ yields a good approximation to $\log O_n$, where

$$T_n := \prod_{r=1}^{n} r^{K_{n,r}} = \prod_{k=1}^{K_n} Z_{n,k},$$

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For Ewens' permutations quite general separable statistics were studied in a series of papers by Babu and Manstavičius, for instance, Babu and Manstavičius (2002), Manstavičius (2009).

Extension of the Erdős-Turán law:

Weak convergence of the product of cycle lengths

$\mu = \mathbb{E}|\log W|$, $\sigma^2 = \mathrm{Var}\,(\log W)$

Theorem (a) If $\sigma^2 < \infty$ then, with

$$b_n = \mu^{-1}\left(2^{-1}\log^2 n - \int_0^{\log n}\int_0^z \mathbb{P}\{|\log(1-W)| > x\}\mathrm{d}x\mathrm{d}z\right) \quad (1)$$

and $a_n = ((3\mu^3)^{-1}\sigma^2 \log^3 n)^{1/2}$, the limiting distribution of $(\log T_n - b_n)/a_n$ is standard normal.

(b) If $\sigma^2 = \infty$, and

$$\int_0^x y^2\, \mathbb{P}\{|\log W| \in \mathrm{d}y\} \ \sim \ \ell(x), \quad x \to \infty,$$

for some ℓ slowly varying at ∞, then, with b_n given by (1) and $a_n = (3\mu^3)^{-1/2} c_{[\log n]} \log n$, where (c_n) is any positive sequence satisfying $\lim_{n\to\infty} n\ell(c_n)/c_n^2 = 1$, the limiting distribution of $(\log T_n - b_n)/a_n$ is standard normal.

Extension of the Erdős-Turán law:
Weak convergence of the product of cycle lengths

$\mu = \mathbb{E}|\log W|$

(c) If

$$\mathbb{P}\{|\log W| > x\} \sim x^{-\alpha}\ell(x), \quad x \to \infty,$$

for some ℓ slowly varying at ∞ and $\alpha \in (1,2)$ then, with

$$b_n = \mu^{-1}\left(2^{-1}\log^2 n - \int_0^{\log n}\int_0^z \mathbb{P}\{|\log(1-W)| > x\}\mathrm{d}x\mathrm{d}z\right)$$

and

$$a_n = ((\alpha+1)\mu^{\alpha+1})^{-1/\alpha} c_{\lfloor \log n \rfloor} \log n,$$

where (c_n) is any positive sequence satisfying $\lim_{n\to\infty} n\ell(c_n)/c_n^\alpha = 1$, the limiting distribution of $(\log T_n - b_n)/a_n$ is the α-stable law with characteristic function

$$u \mapsto \exp\{-|u|^\alpha \Gamma(1-\alpha)(\cos(\pi\alpha/2) + i\sin(\pi\alpha/2)\,\mathrm{sgn}(u))\}, \ u \in \mathbb{R}.$$

Extension of the Erdős-Turán law:
Weak convergence of the product of cycle lengths

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Example 1. Assume that

$$W \overset{d}{=} \mathrm{beta}(\theta, 1), \quad \theta > 0.$$

Then $\mu = \theta^{-1}$, $\sigma^2 = \theta^{-2}$ and

$$\lim_{n\to\infty} \frac{\int_0^{\log n}\int_0^z \mathbb{P}\{|\log(1-W)| > x\}\mathrm{d}x\mathrm{d}z}{\log^{3/2} n} = 0,$$

since the numerator is $O(\log n)$. Application of the theorem yields

$$\frac{\log T_n - (\theta/2)\log^2 n}{\sqrt{(\theta/3)\log^3 n}} \overset{d}{\to} \mathcal{N}(0,1), \quad n\to\infty,$$

which agrees with Arratia & Tavaré (1992).

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and $a_n = ((3\mu^3)^{-1}\sigma^2 \log^3 n)^{1/2}$, the limiting distribution of $(\log T_n - b_n)/a_n$ is standard normal.

Example 2. Assume that, for some $\gamma \in (0, 1/2)$,

$$\mathbb{P}\{W > x\} = \frac{1}{1 + |\log(1-x)|^\gamma}, \quad x \in [0, 1).$$

Then $\mathbb{E}\log^2 W < \infty$ and

$$\mathbb{P}\{|\log(1-W)| > x\} \sim x^{-\gamma} \quad \text{as } x \to \infty,$$

and in this case,

$$a_n = \text{const}\log^{3/2} n$$

and

$$b_n = \mu^{-1}(2^{-1}\log^2 n - \text{const}\log^{2-\gamma} n + o(\log^{2-\gamma} n)).$$

!! Thus we see that the second term $b_n - (2\mu)^{-1}\log n$ of centering cannot be ignored.

Extension of the Erdős-Turán law:
Asymptotics of the cycle partition

Q–a multiplicative random walk with points

$$Q_0 := 1, \quad Q_j := \prod_{i=1}^{j} W_i, \quad j \in \mathbb{N},$$

where (W_i) are independent replicas of W.

$U_1, \ldots, U_n$– a sample from the uniform [0, 1] distribution, independent of Q.

$K_{n,r}$– the number of boxes occupied by exactly r out of the first n balls.

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The cycle partition $(K_{n,1}, \ldots, K_{n,n})$ will remain the same when defined in terms of

$$nQ \quad \text{and} \quad \mathcal{U}_n := \{nU_{j,n} : \ 1 \le j \le n\},$$

where $U_{1,n} \le U_{2,n} \le \ldots \le U_{n,n}$ are the corresponding order statistics.

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$\lim_{n\to\infty} \mathcal{U}_n = \mathcal{U}$ vaguely, where $\mathcal{U}$ is a Poisson process on $\mathbb{R}^+$ with unit intensity.

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For $\mathcal{U}$ a Poisson process on $\mathbb{R}^+$ with unit intensity,

$$\lim_{n\to\infty} \mathcal{U}_n = \mathcal{U} \quad \text{vaguely.}$$

For $-\log \mathcal{Q}$ a stationary renewal process on $\mathbb{R}$,

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Limiting 'balls-in-boxes' scheme:

- 'boxes'–intervals induced by successive points of $\mathcal{Q}$
- 'balls'–points of $\mathcal{U}$

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In the limiting scheme, for $r \in \mathbb{N}$ denote by K_r the number of intervals that contain exactly r Poisson points.

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Theorem (Gnedin, I. and Roesler (2008)). As $n \to \infty$, we have

$$(K_{n,1}, K_{n,2} \dots) \to_d (K_1, K_2, \dots),$$

along with

$$\mathbb{E}[K_{n,r}] \to \mathbb{E}[K_r] = \frac{1}{r\mathbb{E}|\log W|}.$$