On Another Unsolved Problem of Herbert Wilf

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Dedicated to the memory of Philippe Flajolet.

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Introduction

The webpage of Herbert Wilf has 8 Unsolved Problems.
www.math.upenn.edu/~wilf/website/UnsolvedProblems.pdf

Much thanks to Herb Wilf for posing these fascinating questions on:

1. series for π,
2. growth of partition functions,
3. a problem in asymptotics, Resolved by MDW in 2010.
4. the quadratic character of binomial coefficients,
5. Young tableaux,
6. distinct multiplicities (among parts of partitions),
7. Toeplitz determinants, and
8. chromatic number. Progress made by Xuding Zhu in 2010.

MDW is currently working on #6.

Wilf's Sixth Unsolved Problem

Distinct multiplicities

"Let $T(n)$ be the set of partitions of n for which the (nonzero) multiplicities of its parts are all different, and write $f(n) = |T(n)|$.

See Sloane's sequence `A098859` for a table of values.

Find any interesting theorems about $f(n)$.

The mapping that sends a partition of n to another partition of n in which the roles of parts and multiplicities are interchanged is a well defined involution on $T(n)$, which is how I arrived at the study of this problem."

Wilf Partitions: the Nonzero Multiplicities are Distinct

Example: 7 Wilf partitions of $n = 6$; listed with their inversions.

Wilf partitions:

1. $4 + 1 + 1 = (1)4 + (2)1$
2. $3 + 3 = (2)3$
3. $6 = (1)6$
4. $3+1+1+1 = (1)3+(3)1$ (self inverting)

Involutions—also Wilf partitions:

5. $(4)1 + (1)2 = 1 + 1 + 1 + 1 + 2$
6. $(3)2 = 2 + 2 + 2$
7. $(6)1 = 1 + 1 + 1 + 1 + 1 + 1$

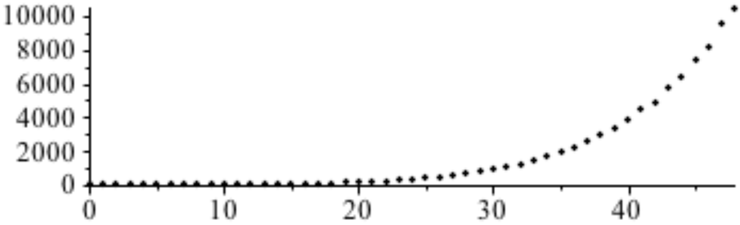
NOT Wilf partitions:

8. $5 + 1 = (1)5 + (1)1$ [“1” is used twice as a multiplicity]
9. $4 + 2 = (1)4 + (1)2$ [“1” is used twice as a multiplicity]
10. $3 + 2 + 1 = (1)3 + (1)2 + (1)1$ [“1” is a triple multiplicity]
11. $2 + 2 + 1 + 1 = (2)2 + (2)1$ [“1”, “2” are used twice as multiplicities]

What was known?

We use a_n as the number of Wilf partitions of n.

The values $a_0, \ldots, a_{48}$ were computed by David S Newman and Vladeta Jovovic for Sloane's database. (Newman said his computer was unable to handle much beyond this.)



No other references for the enumeration of Wilf partitions are listed on Sloane!

Strange: because many questions about partitions were long-ago resolved.

Computing Examples

Newman and Jovovic emailed to say their computation of $a_0, \ldots, a_{48}$ was a filtering of Wilf partitions from the set of all partitions.

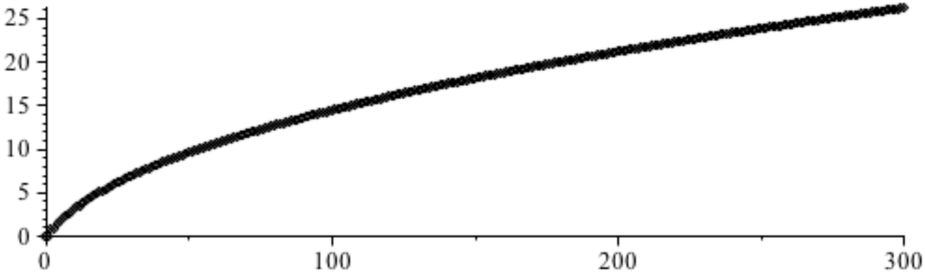
Unfortunately, for instance, there are only 10,434 Wilf partitions of the integer $n = 48$, but there are 147,273 integer partitions altogether of $n = 48$. The situation only gets worse for large n.

MDW used a "bootstrap" method: new Wilf partitions are generated from the previous Wilf partition by adding new "parts" with previously-unused "multiplicities" to the current partition.

By restricted depth-first search, we only touch each Wilf partition once. We only need to keep track of one Wilf partition at a time—memory use is trivial. Execution is proportional to the number of Wilf partitions. Non-Wilf partitions are ignored!

Computing Examples

By this bootstrap method, we compute $a_0, \ldots, a_{300}$. This graph shows the values of $\log(a_n)$ for $0 \le n \le 300$.

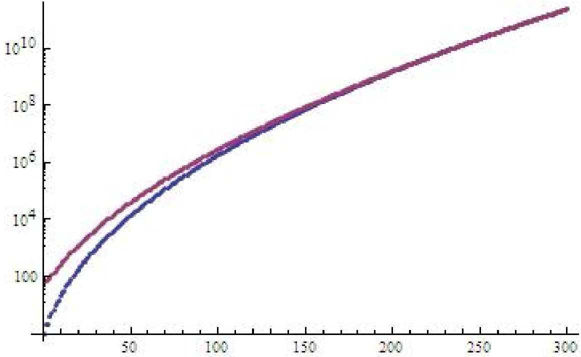


The bootstrapping was absolutely necessary. The number of Wilf partitions is a very small fraction of the total number of integer partitions. For $n = 300$, this fraction is:

$$2.28 \times 10^{11}/(9.25 \times 10^{15}) = 0.0000246$$

Conjecture from Jacek Cichoń

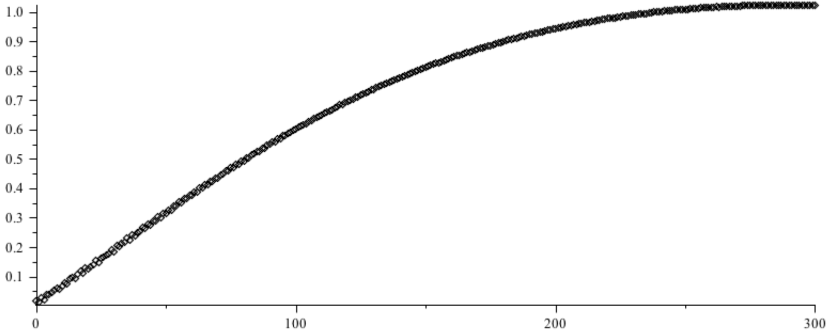
Jacek made a conjecture this spring when he saw my data for $a_0, \ldots, a_{300}$. He suggested that perhaps the leading term of a_n could be $(12.40\ldots/n)\exp(1.69..\sqrt{n})$. His plot is given below.



The blue (lower) numbers are the enumeration of $\log_{10}(a_n)$, i.e., the logarithm of the number of Wilf partitions of n. The purple (upper) numbers are Jacek's proposed approximation.

Conjecture from Jacek Cichoń

Taking the ratio of a_n to Jacek's conjectured $(12.40\ldots/n)\exp(1.69..\sqrt{n})$, we plot the ratio of the actual values to the conjectured values:



Enumerating the Wilf partitions (One Part)

In 2010, Wilf himself said he did not know the generating function for the number of all Wilf partitions. So this is a natural starting point.

Every partition with exactly one part qualifies as a Wilf partition, i.e., any number of the form $n = m_1 p_1$ is a Wilf partition. So the generating function of all Wilf partitions that have *exactly one part* is

$$\sum_{p_1 \geq 1} \sum_{m_1 \geq 1} 1 z^{m_1 p_1} = \sum_{p_1 \geq 1} \frac{z^{p_1}}{1 - z^{p_1}}$$
$$= z + 2z^2 + 2z^3 + 3z^4 + 2z^5 + 4z^6 + 2z^7 + 4z^8 + 3z^9 + 4z^{10} + 2z^{11} + 6z^{12} + 2z^{13} + 4z^{14} + 4z^{15} + 5z^{16} + 2z^{17} + 6z^{18} + 2z^{19} + 6z^{20} + 4z^{21} + 4z^{22} + 2z^{23} \cdots$$

Of course, the coefficient of z^n is the number of divisors of n.

Enumerating the Wilf partitions (Two Parts)

Every partition with exactly two parts qualifies as a Wilf partition, *unless* the multiplicities of the two parts are the same.

I.e., any $n = m_1 p_1 + m_2 p_2$ $(p_1 < p_2)$ is a Wilf partition, unless $m_1 = m_2$. So the GF of all Wilf partitions that have *exactly two parts* is

$$\sum_{p_1 < p_2} \sum_{(m_1, m_2)} 1 z^{m_1 p_1 + m_2 p_2} = \sum_{p_1 < p_2} \left(\frac{z^{p_1}}{1 - z^{p_1}} \frac{z^{p_2}}{1 - z^{p_2}} - \frac{z^{p_1} z^{p_2}}{1 - z^{p_1} z^{p_2}} \right)$$

$$= z^4 + 3z^5 + 3z^6 + 8z^7 + 9z^8 + 12z^9 + 16z^{10} + 22z^{11} + 20z^{12} + 31z^{13} + 35z^{14} + 34z^{15} + 44z^{16} + 51z^{17} + 53z^{18} + 62z^{19} + 65z^{20} \cdots$$

We see the start of an inclusion-exclusion here.

Enumerating the Wilf partitions (Three Parts)

We use inclusion-exclusion to get the Wilf partitions with *exactly three parts*. I.e., any $n = m_1 p_1 + m_2 p_2 + m_3 p_3$ (with p_1, p_2, p_3 distinct) is a Wilf partition, unless $m_1 = m_2$ or $m_1 = m_3$ or $m_2 = m_3$ or $m_1 = m_2 = m_3$. So the GF of all Wilf partitions that have exactly three parts is

$$\sum_{p_1 < p_2 < p_3} \sum_{(m_1, m_2, m_3)} 1 z^{m_1 p_1 + m_2 p_2 + m_3 p_3} = \sum_{p_1 < p_2 < p_3} \left(\frac{z^{p_1}}{1 - z^{p_1}} \frac{z^{p_2}}{1 - z^{p_2}} \frac{z^{p_3}}{1 - z^{p_3}} \right.$$
$$- \frac{z^{p_1} z^{p_2}}{1 - z^{p_1} z^{p_2}} \frac{z^{p_3}}{1 - z^{p_3}} - \frac{z^{p_1} z^{p_3}}{1 - z^{p_1} z^{p_3}} \frac{z^{p_2}}{1 - z^{p_2}} - \frac{z^{p_2} z^{p_3}}{1 - z^{p_2} z^{p_3}} \frac{z^{p_1}}{1 - z^{p_1}}$$
$$\left. + 2 \frac{z^{p_1} z^{p_2} z^{p_3}}{1 - z^{p_1} z^{p_2} z^{p_3}} \right)$$
$$= z^{10} + 4z^{11} + 5z^{12} + 12z^{13} + 16z^{14} + 24z^{15} + 33z^{16} + 52z^{17} + 57z^{18}$$
$$+ 89z^{19} + 100z^{20} + 131z^{21} + 152z^{22} + 207z^{23} + 215z^{24} + \cdots$$

Enumerating the Wilf partitions (Four Parts)

The GF of all partitions that have *exactly four parts* is

$$\sum_{p_1<\cdots<p_4}\sum_{(m_1,\dots,m_4)} 1z^{\sum_j m_j p_j} = \sum_{p_1<\cdots<p_4}\Bigg(\frac{z^{p_1}}{1-z^{p_1}}\frac{z^{p_2}}{1-z^{p_2}}\frac{z^{p_3}}{1-z^{p_3}}\frac{z^{p_4}}{1-z^{p_4}}$$

$$-\frac{z^{p_1}z^{p_2}}{1-z^{p_1}z^{p_2}}\frac{z^{p_3}}{1-z^{p_3}}\frac{z^{p_4}}{1-z^{p_4}} - \frac{z^{p_1}z^{p_3}}{1-z^{p_1}z^{p_3}}\frac{z^{p_2}}{1-z^{p_2}}\frac{z^{p_4}}{1-z^{p_4}} - \frac{z^{p_1}z^{p_4}}{1-z^{p_1}z^{p_4}}\frac{z^{p}}{1-}$$

$$-\frac{z^{p_2}z^{p_3}}{1-z^{p_2}z^{p_3}}\frac{z^{p_1}}{1-z^{p_1}}\frac{z^{p_4}}{1-z^{p_4}} - \frac{z^{p_2}z^{p_4}}{1-z^{p_2}z^{p_4}}\frac{z^{p_1}}{1-z^{p_1}}\frac{z^{p_3}}{1-z^{p_3}} - \frac{z^{p_3}z^{p_4}}{1-z^{p_3}z^{p_4}}\frac{z^{p}}{1-}$$

$$+2\frac{z^{p_1}z^{p_2}z^{p_3}}{1-z^{p_1}z^{p_2}z^{p_3}}\frac{z^{p_4}}{1-z^{p_4}} + 2\frac{z^{p_1}z^{p_2}z^{p_4}}{1-z^{p_1}z^{p_2}z^{p_4}}\frac{z^{p_3}}{1-z^{p_3}} + 2\frac{z^{p_1}z^{p_3}z^{p_4}}{1-z^{p_1}z^{p_3}z^{p_4}}\frac{z^{p_2}}{1-z^{p_2}}$$

$$+1\frac{z^{p_1}z^{p_2}}{1-z^{p_1}z^{p_2}}\frac{z^{p_3}z^{p_4}}{1-z^{p_3}z^{p_4}} + 1\frac{z^{p_1}z^{p_3}}{1-z^{p_1}z^{p_3}}\frac{z^{p_2}z^{p_4}}{1-z^{p_2}z^{p_4}} + 1\frac{z^{p_1}z^{p_4}}{1-z^{p_1}z^{p_4}}\frac{z^{p_2}z^{p_3}}{1-z^{p_2}z^{p_3}}$$

$$-6\frac{z^{p_1}z^{p_2}z^{p_3}z^{p_4}}{1-z^{p_1}z^{p_2}z^{p_3}z^{p_4}}\Bigg)$$

WHOA!!!

$$= z^{20} + 5z^{21} + 8z^{22} + 17z^{23} + 29z^{24} + \cdots$$

Enumerating the Wilf partitions (Four Parts), Revised

The GF of all partitions that have *exactly four parts* is

$$\sum_{p_1<\cdots<p_4}\sum_{(m_1,\ldots,m_4)} 1z^{\sum_j m_j p_j} = \sum_{\{\{p_1\},\ldots,\{p_4\}\}} \frac{z^{p_1}}{1-z^{p_1}}\frac{z^{p_2}}{1-z^{p_2}}\frac{z^{p_3}}{1-z^{p_3}}\frac{z^{p_4}}{1-z^{p_4}}$$

$$+\sum_{\{\{p_1<p_2\},\{p_3\},\{p_4\}\}}(-1)\frac{z^{p_1}z^{p_2}}{1-z^{p_1}z^{p_2}}\frac{z^{p_3}}{1-z^{p_3}}\frac{z^{p_4}}{1-z^{p_4}}$$

$$+\sum_{\{\{p_1<p_2<p_3\},\{p_4\}\}}2\frac{z^{p_1}z^{p_2}z^{p_3}}{1-z^{p_1}z^{p_2}z^{p_3}}\frac{z^{p_4}}{1-z^{p_4}}$$

$$+\sum_{\{\{p_1<p_2\},\{p_3<p_4\}\}}1\frac{z^{p_1}z^{p_2}}{1-z^{p_1}z^{p_2}}\frac{z^{p_3}z^{p_4}}{1-z^{p_3}z^{p_4}}$$

$$\sum_{\{p_1<\cdots<p_4\}}(-6)\frac{z^{p_1}z^{p_2}z^{p_3}z^{p_4}}{1-z^{p_1}z^{p_2}z^{p_3}z^{p_4}}$$

BETTER!!!

All p_j's are also distinct, so we can order each collection by first member e.g., $\{\{3<7\},\{2<5\}\}$ can be rewritten as $\{\{2<5\},\{3<7\}\}$.

If we continue the reasoning above, the coefficient to be placed in front of a term of the form

$$\frac{z^{p_{11}}z^{p_{12}}\cdots z^{p_{1r_1}}}{1-z^{p_{11}}z^{p_{12}}\cdots z^{p_{1r_1}}}\frac{z^{p_{21}}z^{p_{22}}\cdots z^{p_{2r_2}}}{1-z^{p_{21}}z^{p_{22}}\cdots z^{p_{2r_2}}}\cdots\frac{z^{p_{j1}}z^{p_{j2}}\cdots z^{p_{jr_j}}}{1-z^{p_{j1}}z^{p_{j2}}\cdots z^{p_{jr_j}}}$$

should be exactly

$$(-1)^{r_1-1}(-1)^{r_2-1}(-1)^{r_3-1}\cdots(-1)^{r_j-1}(r_1-1)!(r_2-1)!(r_3-1)!\cdots(r_j-1)!$$

Combining with the idea to group p_{ij}'s as sets of nonempty increasing sets, we now get a very succinct representation:

Enumerating the Wilf partitions (Exact Representation)

Now we represent a collection of j nonempty (ordered) sets as

$$\mathbf{p} = \{\{p_{11} < p_{12} < \cdots < p_{1r_1}\}, \ldots, \{p_{j1} < p_{j2} < \cdots < p_{jr_j}\}\}$$

All p_{ab}'s are distinct, and $p_{11} < p_{21} < \cdots < p_{j1}$.

The "j" here depends on the choice of $\mathbf{p}$. *The "j" is exactly the number of "parts" of the Wilf partitions.*

This gives a succinct representation for the OGF of all Wilf partitions:

$$g(z) = \sum_{n \geq 0} a_n z^n = \sum_{\mathbf{p}} \prod_{k=1}^{j} (-1)^{r_k - 1}(r_k - 1)! \frac{z^{p_{k1}} z^{p_{k2}} \ldots z^{p_{kr_k}}}{1 - z^{p_{k1}} z^{p_{k2}} \ldots z^{p_{kr_k}}}.$$

Request #1: This is the best known, but: Can someone make it more "flaj"??? …e.g., wrap inclusion-exclusion into a more succinct form???

Toward an Asymptotic Analysis

H.K. Hwang and I discussed the similarities to the analysis by Hardy & Ramanujan—and, later, Rademacher—which is discussed in the FS *Analytic Combinatorics* book.

The OGF $g(z)$ for the Wilf partitions cannot be extended analytically beyond the unit circle, because of the behavior at each of the roots of unity.

Visual depictions of approximations to the OGF $g(z)$ reinforce the belief that the contribution from $z = 1$ is the largest contribution to the asymptotics, and thus a saddle point approach might be appropriate.

Review: Saddle Point for All Partitions

Recall: The OGF

$$P(z) = \prod_{j\geq 1} \frac{1}{1-z^j}$$

for the set of all integer partitions also has singularities at all of the roots of unity.

In preparation for the saddle point analysis, we write:

$$\begin{aligned} P(z) &= \prod_{j\geq 1} \frac{1}{1-z^j} \\ &= \exp\left(\sum_{j\geq 1} \frac{1}{j}\frac{z^j}{1-z^j}\right) \\ &= \exp\left(\left(\frac{1}{1-z}\right)\left(\frac{z}{1} + \frac{z^2}{2(1+z)} + \frac{z^3}{3(1+z+z^2)} + \cdots\right)\right) \end{aligned}$$

Review: Saddle Point for All Partitions

So, for real $x \to 1^-$, the OGF for the set of ALL integer partitions is

$$P(x) = \exp\left(\left(\frac{1}{1-x}\right)\left(\frac{x}{1} + \frac{x^2}{2(1+x)} + \frac{x^3}{3(1+x+x^2)} + \cdots\right)\right)$$
$$= \exp\left(\left(\frac{1}{1-x}\right)\frac{\pi^2}{6}(1+o(1))\right)$$

Then we solve the saddle point equation and obtain $\zeta(n) = 1 - \frac{\pi}{\sqrt{6n}}$.

The Cauchy integral around the circle of radius $\zeta(n)$, split into a central region (yielding the accurate approximation) and the non-central region (which is bounded and shown to be exponentially small), yields the familiar result that the number of all integer partitions is

$$[z^n]P(z) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}}.$$

Meinardus Method for Asymptotics of Types of Partitions

Asymptotics for other types of partitions can be resolved using saddle point too, e.g., partitions of

1. all integers
2. all distinct integers
3. powers of 2
4. primes
5. squares
6. cubes
7. triangular numbers
8. odd summands

and the list goes on and on, enabled by the Meinardus method (see p. 578–580 of FS).

Unfortunately, the Meinardus method assumes that the partitions to be studied have the form

$$\prod_{m\geq 1}(1-z^m)^{-b_m},$$

and then relies on properties of the Dirichlet series

$$\alpha(s)=\sum_{m\geq 1}\frac{b_m}{m^s}.$$

Unfortunately, the compact representation we have for the Wilf partitions has a very different form, with a sum over all partitions at the front:

$$g(z)=\sum_{n\geq 0}a_n z^n=\sum_{\mathbf{p}}\prod_{k=1}^{j}(-1)^{r_k-1}(r_k-1)!\,\frac{z^{p_{k1}}z^{p_{k2}}\cdots z^{p_{kr_k}}}{1-z^{p_{k1}}z^{p_{k2}}\cdots z^{p_{kr_k}}}.$$

so the Meinardus method does not appear applicable.

Why is the Wilf Problem So Difficult?

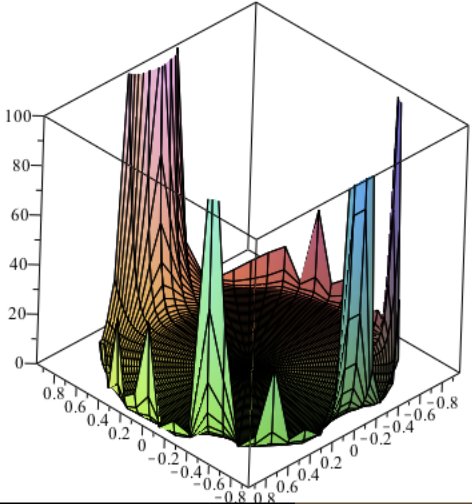
Simply put, the Wilf partitions seem much harder to enumerate because of the relationship of the multiplicities of one "part" to the multiplicities of the other parts.

It is not enough to precisely characterize the parts separately. We must precisely characterize the relationship between the parts.

This makes the analysis quite difficult—but a solution could have enough "surprise" to make it very "flaj"! (OK, sorry.)

Why the Wilf Problem Might NOT be so difficult!

Visualizations of the OGF $g(z) = \sum_{n \geq 0} a_n z^n$ lend credibility to the potential use of the saddle point method.... Key point: the point $z = 1$ should provide the major contribution to the asymptotics of a_n.



Here is an approximate visualization of the OGF for the Wilf partitions with exactly three parts.

Question: How to justify that $z = 1$ provides the main contribution?

Where Does the Saddle Point Method tell us to Integrate?

The OGF $g(z)$ for the Wilf partitions has alternating terms that include $\frac{z^a}{1-z^a}$, where the a is just shorthand for some sums of p_{ij}'s. If we solve the saddle point equation

$$\frac{d}{dx}\left(\frac{z^a}{1-z^a}\frac{1}{z^{n+1}}\right) = 0,$$

we are advised to integrate at radius

$$\zeta(n) = 1 - \frac{1}{n} + \Theta\left(\frac{1}{n^2}\right).$$

Actually, however, we have an alternating sum, with products that include factorials and also several terms of this form. So it is still unclear where the saddle point equation has a true solution.

Request #2: It would be helpful to have a collaboration for conducting the saddle point analysis in a very "flaj" way.