# UNIVERSAL EXPONENTS AND TAIL ESTIMATES IN THE ENUMERATION OF PLANAR MAPS

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# Contents

# One Functional Equation

**(Folkore) Theorem** [Bender, Canfield, Meir+Moon, ...]

Suppose that where $\Phi(z,y)$ has a power series expansion at $(0,0)$ with **non-negative coefficients** and $\Phi_{yy}(z,y) \ne 0$ (+ *minor* technical conditions) and suppose that a power series $y(z) = \sum y_n z^n$ satisfies the functional equation

$$\boxed{y(z) = \Phi(z, y(z))}$$

Let $y_0 > 0$ and $z_0 > 0$ be defined by

$$y_0 = \Phi(z_0, y_0), \quad 1 = \Phi_y(z_0, y_0).$$

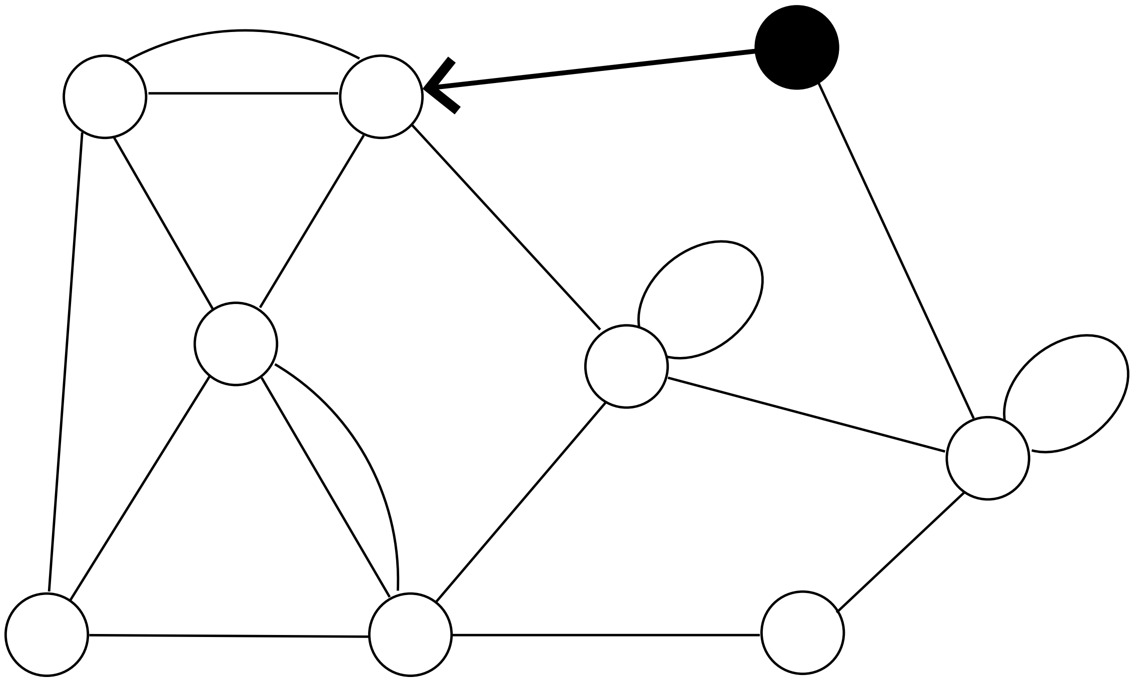
Then there exists analytic function $g(z), h(z) \ne 0$ such that locally

$$\boxed{y(z) = g(z) - h(z)\sqrt{1 - \frac{z}{z_0}}.}$$

Consequently we have (for some constant $c > 0$)

$$\boxed{y_n = [z^n]\, y(z) \sim c \cdot n^{-3/2} z_0^{-n}.}$$

# Enumeration of Planar Maps



A **planar map** is a connected planar graph, possibly with loops and multiple edges, together with an embedding in the plane.
A map is **rooted** if a vertex $v$ and an edge $e$ incident with $v$ are distinguished, and are called the root-vertex and root-edge, respectively. The face to the right of e is called the root-face and is usually taken as the **outer face**.

# Enumeration of Planar Maps

$M_n$ ... number of rooted maps with $n$ edges [Tutte]

$$\boxed{M_n = \frac{2(2n)!}{(n+2)!n!}3^n}$$

The proof is given with the help of generating functions and the so-called **quadratic method**.

Asymptotics:

$$M_n \sim c \cdot n^{-5/2} 12^n$$

# Enumeration of Planar Maps

## Root degree distribution

$d_{n,k}$ ... probability that the root-vertex has degree $k$ in a map with $n$ edges ( $=$ probability that the outer face has valency $k$ in a map with $n$ edges – by duality)

$d_k = \lim_{n\to\infty} d_{n,k}$ :

$$\boxed{d_k \sim c \cdot k^{1/2} \left(\frac{5}{6}\right)^k}$$

# Enumeration of Planar Maps

**Universal Property**

$$M_n \sim c \cdot n^{-5/2} \gamma^n, \quad d_k \sim c \cdot k^{1/2} q^k$$

| Class of maps | $\gamma$ | $q$ |
|---|---|---|
| Arbitrary | 12 | 5/6 |
| Eulerian | 8 | $\sqrt{3}/2$ |
| 3-connected | 4 | 1/2 |
| Loopless | 256/27 | 3/4 |
| 2-connected | 27/4 | 2/3 |
| Bipartite | 8 | 3/4 |

[Banderier, Flajolet, Schaeffer, Soria] (for the $n^{-5/2}$-property):
'This generic asymptotic form is "universal" in so far as it is valid for all known "natural families of maps".'

**Problem.** Is there a reason for this **universal property**?

# Enumeration of Planar Maps

## Generating functions

$M_{n,k}$ ... number of maps with $n$ edges and outer-face-valency $k$

$M(z,u) = \sum_{n,k} M_{n,k} u^k z^n$

$$M(z,u) = 1 + zu^2 M(z,u)^2 + uz \frac{uM(z,u) - M(z,1)}{u-1}.$$

$u$ ... **“catalytic variable”**

# The Quadratic Method

Completing the square leads to

$$\boxed{\left[2u^2z(1-u)M(z,u)-(1-u+u^2z)\right]^2 = H(z,u,M(z,1))}$$

with

$$H(z,u,y) = 4(u-1)u^3z^2y + u^4z^2 - 4u^4z + 6u^3z - 2u^2z + u^2 - 2u + 1.$$

**General Form:**

$$\boxed{[G_1(z,u,y(z))M(z,u) + G_2(z,u,y(z))]^2 = H(z,u,y(z))}$$

where $y(z)$ abbreviates $M(z,1)$.

# The Quadratic Method

$$[G_1(z,u,y(z))M(z,u)+G_2(z,u,y(z))]^2 = H(z,u,y(z))$$

**Key observation:**

$$\exists\, u(z) \text{ with } H(z,u(z),y(z))=0 \quad \Longrightarrow \quad H_u(z,u(z),y(z))=0$$

**Quadratic Method**

1. Solve the system $\boxed{H(z,u(z),y(z))=0, \quad H_u(z,u(z),y(z))=0}$
2. $M(z,1)=y(z)$
3. $M(z,u)=\left(\sqrt{H(z,u,y(z))}-G_2(z,u,y(z))\right)/G_1(z,u,y(z))$.

# The Quadratic Method

## Planar Maps

**ad 1.** $u = u(z)$ and $y(z) = M(z,1)$ are determined by

$$z = \frac{(1-u)(2u-3)}{u^2}, \quad M(z,1) = -u\frac{3u-4}{(2u-3)^2}$$

**ad 2.** Elimination gives an equation for $M = M(z,1)$:

$$27z^2M^2 - 18zM + M + 16z - 1 = 0$$

and consequently

$$M(z,1) = -\frac{1}{54z^2}\left(1 - 18z - \boxed{(1-12z)^{3/2}}\right) = \sum_{n\ge 0}\frac{2(2n)!}{(n+2)!n!}3^nz^n.$$

**ad 3.**

$$M(z,u) = \frac{\sqrt{H(z,u,M(z,1))} + 1 - u + u^2z}{2u^2z(1-u)}.$$

# The Quadratic Method

## Planar Maps

Local expansions:

$$M(z,1) = y_0 + y_2(1-12z) + y_3(1-12z)^{3/2} + O((1-12z)^2).$$

$$M(z,u) = y_0(u) + y_2(u)(1-12z) + y_3(u)(1-12z)^{3/2} + O((1-12z)^2),$$

$$\Longrightarrow \quad M_n \sim \frac{2}{\sqrt{\pi}} n^{-5/2} 12^n$$

$$\sum_{k\ge 1} d_k u^k = \lim_{n\to\infty} \frac{[z^n]M(z,u)}{[z^n]M(z,1)} = \frac{y_3(u)}{y_3} = \frac{u\sqrt{3}}{\sqrt{(2+u)(6-5u)^3}}$$

$$\Longrightarrow d_k \sim c \cdot k^{1/2} \left(\frac{5}{6}\right)^k$$

# The $(1-z)^{3/2}$-paradigma

## THEOREM 1

Let $F(z,u)=\sum_{n,k} f_{n,k}u^k z^n$ and $y(z)=\sum_n y_n z^n$ with $f_{n,k}\ge 0$ and $y_n \ge 0$ be the unique solutions of the equation

$$[G_1(z,u,y(z))F(z,u)+G_2(z,u,y(z))]^2 = H(z,u,y(z)),$$

(+ *some technical conditions*). Furthermore assume that

$$\boxed{H(z_0,y_0,u_0)=0, \quad H_u(z_0,y_0,u_0)=0, \quad H_{uu}(z_0,y_0,u_0)=0,}$$

$$G_1 \ne 0, \quad H_y \ne 0, \quad H_{uy}\ne 0, \quad H_{uuu}\ne 0, \quad H_z H_{uy} \ne H_y H_{zu},$$

$$H_{uuu}H_{uy}^2 - H_y H_{uy} H_{uuuu} + 3 H_y H_{uuy} H_{uuu} \ne 0$$

Then there exists analytic function $g(z), h(z) \ne 0$ such that locally

$$\boxed{y(z) = g(z) - h(z)\left(1-\frac{z}{z_0}\right)^{3/2}.}$$

Consequently we have (for some constant $c>0$)

$$\boxed{y_n = [z^n]\, y(z) \sim c\cdot n^{-5/2} z_0^{-n}.}$$

# Universal tail estimates

**THEOREM 1 (cont.)**

Furthermore we have that for every integer $k \geq 0$ the limit

$$d_k = \lim_{n \to \infty} \frac{f_{n,k}}{y_n}$$

exists and we have, uniformly for $k \leq C \log n$,

$$\frac{f_{n,k}}{d_n} \sim \overline{c} \cdot q^k k^{1/2}$$

for some $\overline{c} > 0$, $q = 1/u_0$ and any constant $C > 0$, in particular

$$\boxed{d_k \sim \overline{c} \cdot k^{1/2} q^k.}$$

# Applications

## Planar Maps

$$\boxed{M(z,u) = 1 + zu^2 M(z,u)^2 + uz\frac{uM(z,u) - M(z,1)}{u-1}.}$$

$$H(z,u,y) = 4(u-1)u^3z^2y + u^4z^2 - 4u^4z + 6u^3z - 2u^2z + u^2 - 2u + 1.$$

$$G_1(z,u) = 2u^2z(1-u), \quad G_2(z,u) = 1 - u + u^2z$$

Critical point: $(z_0, u_0, y_0) = (1/12, 6/5, 4/3)$.

$$G_1 = -\frac{6}{125}, \quad H_y = \frac{6}{625}, \quad H_{uy} = \frac{9}{125}, \quad H_{uuu} = -\frac{50}{9}, \quad H_zH_{uy} - H_yH_{zu} = \frac{288}{15625},$$

$$H_{uuu}H_{uy}^2 - H_yH_{uy}H_{uuuu} + 3H_yH_{uuy}H_{uuu} = \frac{43}{625}.$$

# Applications

## 2-connected maps

$$\boxed{B(z,u)^2 - (u^2 z + uB(z) + 1)B(z,u) + (u^3 + u)B(z,1)z + u^2 z - uz = 0.}$$

$$H(z,u,y) = u^2 y^2 - (2u^3 z + 2u)y + u^4 z^2 - 2u^2 z + 4uz + 1.$$

Critical point: $(z_0, u_0, y_0) = (4/27, 3/2, 4/3)$.

All non-vanishing conditions are satisfied.

# Applications

## Bipartite maps

$$\boxed{B(z,u) = 1 + u^2 z B(z,u)^2 + u^2 z \frac{B(z,u) - B(z,1)}{u^2 - 1}.}$$

$H(z,u,y) = (4u^6z^2 - 4u^4z^2)y + u^4 - 2u^2 + 6u^4z + 1 - 2u^2z + u^4z^2 - 4u^6z.$

Critical point: $(z_0, u_0, y_0) = (1/8, 2/\sqrt{3}, 5/4)$.
All non-vanishing conditions are satisfied.

# Applications

**Eulerian maps**
are dual to bipartite maps.

Furthermore the number of bipartite maps of size $n$ whose root-vertex has degree $k$ is the same as the number of bipartite maps of size $n$ whose root-face has degree $2k$. Hence both distributions are essentially the same.

# Applications

## (Near-) Triangulations

A near-triangulation is a map in which all faces are triangles except possibly the outer face.
$T_{n,k}$ ... number of near-triangulations with $n$ internal vertices and $k+3$ external vertices:

$$\boxed{u^2T(z,u)^2 + (z + zuT(z,0) - u - u^2)T(z,u) + u - zT(z,0) = 0.}$$

$$H(z,u,y) = z^2u^2y^2 + 2(z^2u + zu^2 - zu^3)y + (z - u - u^2)^2 - 4u^3.$$

Critical point: $(z_0, u_0, y_0) = (27/256, 3/16, 32/27)$.
All non-vanishing conditions are satisfied.

**Remark.** Finally one has to work with $S(z,u) = T(z,zu)$.

# Applications

## Universal Property

$$M_n \sim c \cdot n^{-5/2}\gamma^n, \quad d_k \sim c \cdot k^{1/2} q^k$$

| Class of maps | $\gamma$ | $q$ |
|---|---|---|
| Arbitrary | 12 | 5/6 |
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**Remark.** The result for the degree distribution in 2-connected and bipartite maps is new.

# The Analytic Quadratic Method

## Lemma 1

Suppose that $z_0, u_0, y_0$ are complex numbers and that $H(z,u,y)$ is a function that is analytic at $(z_0, u_0, y_0)$ and satisfies the properties

$$\boxed{H(z_0,u_0,y_0) = 0, \quad H_u(z_0,u_0,y_0) = 0, \quad H_{uu}(z_0,u_0,y_0) = 0}$$

and (for $(y,z,u) = (y_0,z_0,u_0)$):

$$H_y \neq 0, \quad H_{uy} \neq 0, \quad H_{uuu} \neq 0, \quad H_z H_{uy} \neq H_y H_{uz}$$

Then the system of functional equations

$$\boxed{H(z,u(z),y(z)) = 0, \quad H_u(z,u(z),y(z)) = 0}$$

has precisely two solutions $u(z)$ and $y(z)$ with $u(z_0) = u_0$ and $y(z_0) = y_0$ which are given by

$$u(z) = g_1(z) \pm g_2(z)\sqrt{1 - \frac{z}{z_0}},$$

$$y(z) = h_1(z) \pm h_2(z)\left(1 - \frac{z}{z_0}\right)^{3/2}$$

in a neighbourhood of $z_0$.

# The Analytic Quadratic Method

## Lemma 1 (cont.)

The functions $g_1(z)$, $g_2(z)$, $h_1(z)$, and $h_2(z)$ are analytic at $z_0$ and satisfy

$$
\begin{aligned}
g_1(z_0) &= u_0, \\
g_2(z_0) &= \sqrt{\frac{2z_0(H_y H_{uz} - H_z H_{uy})}{H_y H_{uuu}}} \neq 0, \\
h_1(z_0) &= y_0, \\
h_2(z_0) &= g_2(z_0)\frac{2z_0(H_y H_{uz} - H_z H_{uy})(H_{uuu}H_{uy}^2 - H_y H_{uy} H_{uuuu} + 3H_y H_{uuy} H_{uuu})}{3H_{uuu}H_{uy}^2 H_y^2}
\end{aligned}
$$

(and all derivatives of $H$ have to be evaluated at $(z,u,y) = (z_0,u_0,y_0)$, respectively).

# The Analytic Quadratic Method

## Proof steps

1. $H_u(z,u,y) = 0 \quad \Longrightarrow \quad y = Y(z,u)$

2. $H(z,u,Y(z,u)) = 0 \quad \Longrightarrow \quad u = u(z)$

3. $y(z) = Y(z,u(z))$

# The Analytic Quadratic Method

**ad 1.** $H_u(z,u,Y(z,u)) = 0$

Implicit function theorem $\Longrightarrow Y(z,u)$ analytic at $(z_0,u_0)$ but

$$\boxed{Y_u(z_0,u_0) = -\frac{H_{uu}}{H_{uy}} = 0.}$$

**ad 2.** $H(z,u,Y(z,u)) = 0$, $u = u(z)$

Folklore Theorem $\Longrightarrow u(z) = g_1(z) \pm g_2(z)\sqrt{1 - \frac{z}{z_0}}$

# The Analytic Quadratic Method

**ad 3.** $y(z) = Y(z, u(z))$

$Y_u(z_0, u_0) = 0 \Longrightarrow$

$$
\begin{aligned}
y(z) &= Y(z, u(z)) \\
&= y_0 + Y_z(z_0, u_0)(z - z_0) + \frac{1}{2} Y_{uu}(z_0, u_0)(u(z) - u_0)^2 \\
&+ Y_{uz}(z_0, u_0)(z - z_0)(u(z) - u_0) + \frac{1}{6} Y_{uuu}(z_0, u_0)(u(z) - u_0)^3 \\
&+ O((z - z_0)^2) \\
&= h_1(z) \pm h_2(z) \left(1 - \frac{z}{z_0}\right)^{3/2}
\end{aligned}
$$

# The Analytic Quadratic Method

**Lemma 2**

Suppose that $G(z, y, u)$ is an analytic function and set

$$F(z,u) = \sum_{n,k} f_{n,k} z^n u^k = \sqrt{G(y(z), z, u)},$$

where $y(z) = \sum_{n\ge 0} y_n z^n$ (with $y_n \ge 0$) and can be locally represented by

$$y(z) = h_1(z) + h_2(z) \left(1 - \frac{z}{z_0}\right)^{3/2}$$

with functions $h_1(z)$, $h_2(z)$, that are analytic at $z = z_0$ and satisfy $y_0 = h_1(z_0) \neq 0$ and $h_2(z_0) \neq 0$, and where

$$G = 0, \quad G_u = 0, \quad G_{uu} = 0, \quad G_{uuu} \neq 0, \quad G_y \neq 0$$

evaluated at $(y, z, u) = (z_0, y_0, u_0)$.

# The Analytic Quadratic Method

**Lemma 2 (cont.)**

Then for every integer $k \ge 0$ the limit

$$\boxed{d_k = \lim_{n\to\infty} \frac{f_{n,k}}{y_n}}$$

exists and we have uniformly for $k \le C \log n$

$$\frac{f_{n,k}}{y_n} \sim \overline{c}\, q^k k^{1/2}$$

for some $\overline{c} > 0$ and $q = 1/u_0$ for any constant $C > 0$. In particular we have

$$\boxed{d_k \sim \overline{c}\, q^k k^{1/2}}$$

as $k \to \infty$.

# The Analytic Quadratic Method

## Sketch of the proof

$$
\begin{aligned}
G(z,y(z),u) =& G(z_0,y_0,u) + G_z(z_0,y_0,u)(z-z_0) \\
&+ G_y(z_0,y_0,u)(y(z)-y_0) + O\left(|1-z/z_0|^2\right) \\
=& \frac{1}{6}G_{uuu}(z_0,y_0,u_0)(u-u_0)^3 + O\left(|1-u/u_0|^4\right) \\
&+ \left(G_z(z_0,y_0,u) + G_y(z_0,y_0,u)h_1'(z_0)\right)(z-z_0) \\
&+ G_y(z_0,y_0,u_0)h_2(z_0)(1-z/z_0)^{3/2} \\
&+ O\left(|1-z/z_0|^{3/2}|1-u/u_0| + |1-z/z_0|^2\right) \\
=& \boxed{A(u)(z-z_0) + B(1-z/z_0)^{3/2} + C(u)(1-u/u_0)^3} \\
&+ O\left(|1-z/z_0|^2 + |1-z/z_0|^{3/2}|1-u/u_0|\right)
\end{aligned}
$$

# The Analytic Quadratic Method

$\Longrightarrow$ (if $|u-u_0| \gg |z-z_0|$ !!)

$$
\begin{aligned}
F(z,u) &= \sqrt{G(z,y(z),u)} \\
&= \sqrt{C(u)}\,(1-u/u_0)^{3/2}\left(1+\frac{A(u)(z-z_0)+B(1-z/z_0)^{3/2}}{C(u)(1-u/u_0)^3}\right. \\
&\qquad\qquad \left. + O\left(\frac{|1-z/z_0|^2}{|1-u/u_0|^3}+\frac{|1-z/z_0|^{3/2}}{|1-u/u_0|^2}\right)\right)^{1/2} \\
&= \sqrt{C(u)}\,(1-u/u_0)^{3/2} + \boxed{\frac{A(u)(z-z_0)+B(1-z/z_0)^{3/2}}{2\sqrt{C(u)}\,(1-u/u_0)^{3/2}}} \\
&\quad + O\left(\frac{|1-z/z_0|^2}{|1-u/u_0|^{3/2}}+\frac{|1-z/z_0|^{3/2}}{|1-u/u_0|^{1/2}}\right)
\end{aligned}
$$

+ double Cauchy integration.

# Thank You!