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Disintegration

Fixed points of the smoothing transform: Two-sided solutions

M. Meiners

Uppsala universitet

June 13, 2011

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This talk is based on...

Alsmeyer, G., Meiners, M.

Fixed points of the smoothing transform: Two-sided solutions (2010). *Submitted*

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Alsmeyer, G., Meiners, M.

Fixed points of the smoothing transform: Two-sided solutions (2010). *Submitted*

Alsmeyer, G., Biggins, J. D., Meiners, M.

The functional equation of the smoothing transform (2010). To appear in *Ann. Probab.*

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Set-up

We consider equations of the type

$$X \stackrel{d}{=} \sum_{j \geq 1} T_j X_j + C, \tag{1}$$

where

- $(C, T_1, T_2, \ldots)$ is a given sequence of real-valued random variables, $T_j \geq 0$, $N = \#\{j : T_j > 0\} < \infty$ a.s.;
- $(X_1, X_2, \ldots)$ is a sequence of i.i.d. random variables independent of $(C, T_1, T_2, \ldots)$;
- X has the same distribution as the X_j;
- the distribution of X is considered unknown.

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Equations of stability

One particular instance of (1) are equations of stability:

$$X \stackrel{d}{=} c_n^{-1} \sum_{j=1}^{n} X_j + \gamma_n,$$

(with $c_n = n^{1/\alpha}$, $\gamma_n \in \mathbb{R}$).

Equations of the type

$$X \stackrel{d}{=} \sum_{j \geq 1} T_j X_j + C, \tag{1}$$

frequently arise in the analysis of divide and conquer algorithms.

Rösler, U., Rüschendorf, L.

The contraction method for recursive algorithms.
Algorithmica **29**, 3–33, 2001.

Asymptotic number of comparisons

Theorem (Rösler '91)

After centering and suitable normalization, the (random) number of comparisons the sorting algorithm $\mathit{QUICKSORT}$ *needs to order a list of n elements, converges in distribution as* $n \to \infty$*. The limit law P satisfies*

$$X \overset{d}{=} UX_1 + (1-U)X_2 + C$$

where X, X_1, X_2 *are i.i.d.* $\sim P$ *and U is* $\mathrm{Unif}[0,1]$ *and C is some function of U,* X_1 *and* X_2 *independent of U.*

The solutions to the QUICKSORT equation

Theorem (Fill & Janson '00)

Let P denote the QUICKSORT limit from the last slide. Then the set of solutions to the QUICKSORT equation is given by

$$\{\mathrm{dist}(W^* + Y) : Y \sim \mathscr{C}(\mu, \sigma) \text{ indep. of } W^*\},$$

where $\mathscr{C}(\mu, \sigma)$ denotes the Cauchy distribution with parameters μ and σ, and $W^* \sim P$.

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In order to avoid trivial and simple cases as well as case distinctions, we assume throughout that

(A1) $\mathbb{P}(T_j \in r^{\mathbb{Z}} \cup \{0\} \text{ for all } j \geq 1) < 1 \text{ for all } r > 1.$

(A2) $$\mathbb{E}N = \mathbb{E}\sum_{j\geq 1} \mathbb{1}_{\{T_j>0\}} > 1.$$

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The function m

Let

$$m : [0,\infty) \to [0,\infty], \quad \gamma \mapsto \mathbb{E} \sum_{j=1}^{N} T_j^{\gamma}$$

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m is the Laplace transform of a suitable measure on $\mathbb{R}$.

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By (A2), $m(0) = \mathbb{E}N > 1$.

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By (A2), $m(0) = \mathbb{E}N > 1$.

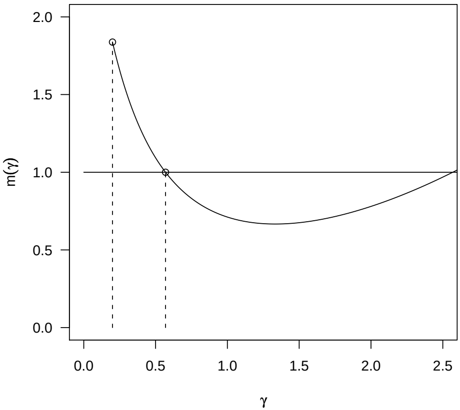
Further, we make the assumptions that

(A3) There is an $\alpha \in (0, 2] : m(\alpha) = 1$;

(A4) There is a $\theta < \alpha : m(\theta) < \infty$.

We assume throughout that α is the minimal such real and call it the *characteristic exponent*.

m and the characteristic exponent



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The main result

Theorem

Suppose that (A1)-(A4) *and some condition on* C *hold. Then there is a coupling* (W^*, W) *such that* W^* *solves* (1), $W \geq 0$ *is a solution to* $W \sim \sum_{j\geq 1} T_j^\alpha W_j$ *and the set of solutions to* (1) *is given by the family*

$$\{\mathrm{dist}(W^* + W^{1/\alpha} Y) : Y \sim \mathscr{S}_\alpha(\sigma, \beta, \mu) \text{ indep. of } (W^*, W)\}$$

where $\mathscr{S}_\alpha(\sigma, \beta, \mu)$ *denotes the stable distribution with index* α*, scale parameter* σ*, skewness parameter* β*, and shift* μ*.*

The main result

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where $\mathscr{S}_\alpha(\sigma, \beta, \mu)$ *denotes the stable distribution with index* α, *scale parameter* σ, *skewness parameter* β, *and shift* μ. *The range of the parameters is given by*

$$(\sigma, \beta, \mu) \in \begin{cases} [0, \infty) \times [-1, 1] \times \{0\} & \text{if } \alpha \notin \{1, 2\}; \\ [0, \infty) \times \{0\} \times \mathbb{R} & \text{if } \alpha = 1; \\ [0, \infty) \times \{0\} \times \{0\} & \text{if } \alpha = 2. \end{cases}$$

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The functional equation

The fixed-point equation

$$X \stackrel{d}{=} \sum_{j \geq 1} T_j X_j + C \tag{1}$$

is equivalent to the functional equation

$$\phi(t) = \mathbb{E} e^{itC} \prod_{j \geq 1} \phi(T_j t) \tag{2}$$

for the characteristic function ϕ of X.

The homogeneous functional equation

The homogeneous fixed-point equation

$$X \stackrel{d}{=} \sum_{j \geq 1} T_j X_j \tag{3}$$

is equivalent to the functional equation

$$\phi(t) = \mathbb{E} \prod_{j \geq 1} \phi(T_j t) \tag{4}$$

for the characteristic function ϕ of X.

The homogeneous functional equation

The homogeneous fixed-point equation

$$X \overset{d}{=} \sum_{j \geq 1} T_j X_j \tag{3}$$

is equivalent to the functional equation

$$\phi(t) = \mathbb{E} \prod_{j \geq 1} \phi(T_j t) \tag{4}$$

for the characteristic function ϕ of X. For the rest of this talk, we will only consider the homogeneous case.

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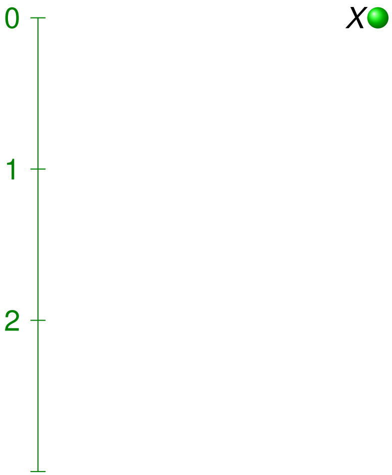
Iteration of (3)



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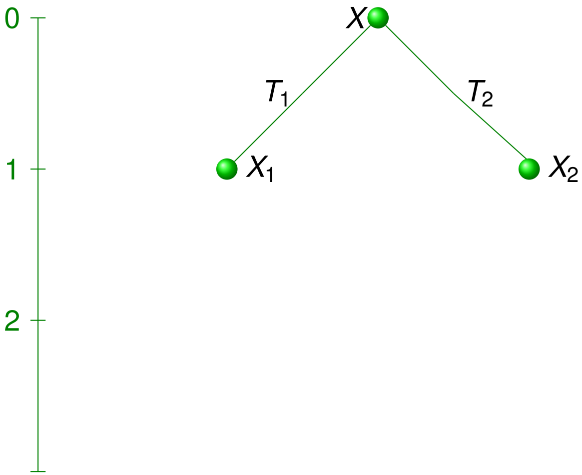

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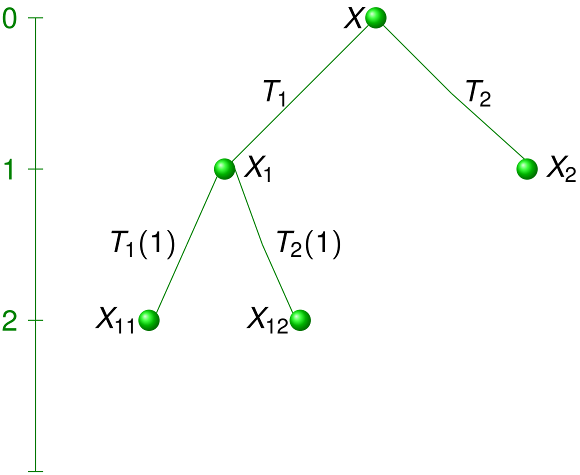

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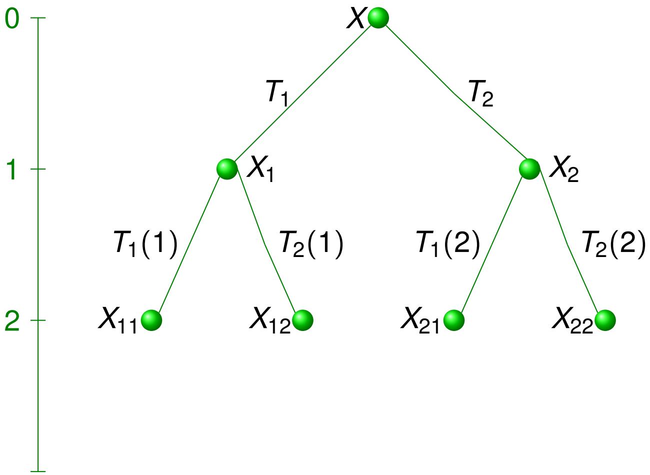

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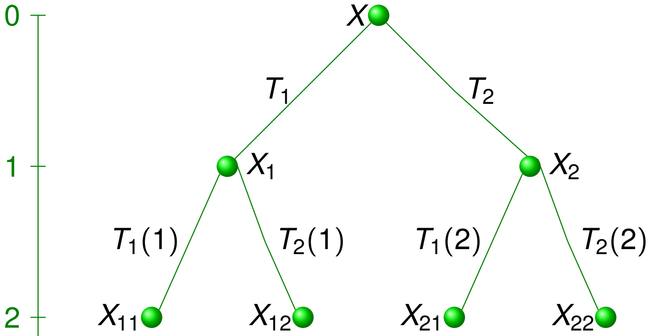

$$L(11)X_{11} + L(12)X_{12} + L(21)X_{21} + L(22)X_{22}$$

The weighted branching process

- Let $\mathbb{V} := \bigcup_{n \geq 0} \mathbb{N}^n$ denote the infinite Ulam-Harris tree.
- Let $(T(v))_{v \in \mathbb{V}}$ be a family of independent copies of $(T_j)_{j \geq 1}$,

$$T(v) = (T_j(v))_{j \geq 1} \stackrel{d}{=} (T_j)_{j \geq 1}.$$

- Let

$$L(\varnothing) := 1 \quad \text{and} \quad L(vj) := L(v) T_j(v)$$

  $(v \in \mathbb{V}, j \in \mathbb{N})$.
- Let $(X(v))_{v \in \mathbb{V}}$ be a sequence of i.i.d. copies of X independent of $(T(v))_{v \in \mathbb{V}}$.

Disintegration

After n iterations, (3) takes the form

$$X \stackrel{\mathrm{d}}{=} \sum_{|v|=n} L(v)X(v).$$

The corresponding functional equation is:

$$\phi(t) = \mathbb{E} \prod_{|v|=n} \phi(L(v)t). \tag{5}$$

Let

$$\Phi_n(t) := \prod_{|v|=n} \phi(L(v)t). \tag{6}$$

Lemma (Biggins/Kyprianou '97)

Let ϕ and Φ_n be as above. Then $(\Phi_n(t))_{n\geq 0}$ constitutes a bounded complex-valued martingale, and thus converges a.s. and in mean to a random variable $\Phi(t)$ satisfying

$$\mathbb{E}\Phi(t) \; = \; \phi(t).$$

The martingales $(\Phi_n(t))_{n\geq 0}$, $t \geq 0$ are called *multiplicative martingales*.

Definition

The process $\Phi = (\Phi(t))_{t\geq 0}$ will be called *disintegration of* ϕ.

A different point of view

$$\Phi_n(t) = \prod_{|v|=n} \phi(L(v)t).$$

can be seen as the characteristic function of $\sum_{|v|=n} L(v)X(v)$ given $(L(v))_{v\in\mathbb{V}}$.

A different point of view

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Given $(L(v))_{v\in\mathbb{V}}$, $(L(v)X(v))_{|v|=n,n\geq 0}$ is a triangular array.

A different point of view

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can be seen as the characteristic function of $\sum_{|v|=n} L(v)X(v)$ given $(L(v))_{v\in\mathbb{V}}$.

Given $(L(v))_{v\in\mathbb{V}}$, $(L(v)X(v))_{|v|=n,n\geq 0}$ is a triangular array.

This triangular array is independent and infinitesimal (for almost all realizations of $(L(v))_{v\in\mathbb{V}}$) by

Theorem (Biggins '98)

In the given situation, $m(\alpha) = 1$ *implies that* $\sup_{|v|=n} L(v) \to 0$ *almost surely as* $n \to \infty$.

Proposition (Caliebe '03)

Almost surely as $n \to \infty$, $(\Phi_n)_{n \geq 0}$ *converges pointwise to a random characteristic function* Φ *of the form* $\Phi = \exp(\Psi)$ *with*

$$\Psi(t) \; = \; \mathrm{i}\, W_1 t - \frac{W_2 t^2}{2} + \int \left(e^{\mathrm{i}\, tx} - 1 - \frac{\mathrm{i}\, tx}{1+x^2} \right) \nu(\mathrm{d}x),$$

where W_1 *and* W_2 *are* $\mathbb{R}$- *and* $[0,\infty)$-*valued* $(L(v))_{v \in \mathbb{V}}$-*measurable random variables, respectively, and* ν *is a (random) Lévy measure such that, for any* $t > 0$, $\nu([t,\infty))$ *and* $\nu((-\infty,-t])$ *are* $(L(v))_{v \in \mathbb{V}}$-*measurable.*

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Proof.

Use the theory of Gnedenko and Kolmogorov. □

From the definition of Ψ, one can derive the following equation for Ψ:

$$\Psi(t) = \sum_{|v|=n} [\Psi]_v(L(v)t) \quad \text{almost surely.}$$

From this pathwise functional equation, one can derive the form of Ψ.

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$$\Psi(t) = \sum_{|v|=n} [\Psi]_v(L(v)t) \quad \text{almost surely.}$$

From this pathwise functional equation, one can derive the form of Ψ.

$$\Psi(t) = \begin{cases} -\sigma^\alpha W|t|^\alpha \left[1 - i\beta \frac{t}{|t|} \tan\left(\frac{\pi\alpha}{2}\right)\right], & \alpha \notin \{1,2\}, \\ i\mu Wt - \sigma W|t|, & \alpha = 1, \\ -\sigma^2 Wt^2, & \alpha = 2. \end{cases}$$

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Thank you for your attention.