Optimal Selection via Dynamic Programming

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Extremely involved . . .
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Numerous improvements; some recent. However:

The optimality of these algorithms is asymptotic.

None is acceptable for small sets.
Sicilian median selection

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The algorithm is deterministic, and one of its versions uses at most $1.5n$ comparisons,

The expected number is $4/3n$.

Extremely efficient, and easy to implement;

but it only approximates the median.
Example
A Sicilian algorithm variant

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For $n$ entries we need work-area of $4\log_3 n$ positions; or $(b + 1) \log_b n$ when using buffers of size $b$. 
Cost of search – in comparisons

What if we use larger groups then $b = 3$, e.g. 5? 7? 9?
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The expected total number of comparisons when looking

in a list of size \( n \) is \( C_b n \), where \( C_b = \frac{V_m(b)}{(b - 1)} \)

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<th>( b )</th>
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Here is why: Say $n = b^r$. We need $n/b$, $n/b^2$, … $n/b^r = 1$ exact medians. The expected cost is

$$n\overline{V}_m(b) \sum_{j=1}^{r} b^{-j} \approx n\overline{V}_m(b)/(b - 1) \overset{\text{def}}{=} nC_b$$
Searching for a selection algorithm

How to do it? How to locate fast—on the average—the median for $b = 5, 7, 9$? nothing much that I could find, except. . .

Knuth provides a table with these numbers, no algorithm or proof, nothing about derivation; but he has some insights:
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Formal derivation needs notation:

\( S \) – the information state about the entries, a digraph. Edge from large to small.

\( U(S) \) – the expected cost to complete the selection, starting at state \( S \).

\( \mathcal{S}_r \) – the set of “resolved” states, which hold enough information to determine the selection.

\( C(S) \) – set of possible actions at state \( S \).
Searching for a selection algorithm

With this notation we have the natural recurrence \((\text{MDP})\)

\[
U(S) = 1 + \min_{\varepsilon \in C(S)} E_{\varepsilon} U(S | \varepsilon), \quad S \not\in S_r,
\]

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\[
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(4)
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\[ U(S) = 1 + \min_{\varepsilon \in C(S)} E_{\varepsilon} U(S \mid \varepsilon), \quad S \notin \mathcal{I}_r, \]  

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This is a Bellman optimality equation. Satisfies the conditions for the final value \( U(S_0) \) to be the optimal cost of resolving \( S_0 \).

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This recurrence is more versatile than may appear.
Optimality in worst-case behavior

If we change the operator $E_\varepsilon$ for $\max_\varepsilon$ we see,

$$U(S) = 1 + \min_{\varepsilon \in C(S)} \max_{\varepsilon} U(S | \varepsilon), \quad S \notin \mathcal{S}_r.$$  \hspace{1cm} (9)

This is a recurrence for the optimal worst-case cost, and policy! The same criteria determine termination.
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Difficulty: Billions-and-billions to choose from!

$$u_n = 3^{n(n-1)/2}; \quad u_8 = 2.28768 \times 10^{13}$$
Schematic description for $K_{4,2}$
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- $a$, $b$
- $c$, $d$

$S_1$

$S_x$: $d:a$, $d:c$, $d:b$
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A general rule: exclude entries known to be too extreme.
Computational process

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\[ U(S_x) = 1.5 \]

\[
\begin{array}{c}
0.5 \\
d : a
\end{array}
\begin{array}{c}
0.75 \\
d : c
\end{array}
\begin{array}{c}
1.67 \\
d : b
\end{array}\]
Computational process, Cont.

(4). A resolved state is marked with the item selected.

(5). Now an entire optimal policy can be captured.

Here is a function to locate \( K_{4,2} \), as generated by the search:

```c
int K42(int *Q) {
    if (Q[0] > Q[1]) swap(&Q[0], &Q[1]); // symmetrization;
    L0: if(Q[2] > Q[3]) goto L1; else goto L2;
    L1: if(Q[0] > Q[3]) goto L3; else goto L4;
    L2: if(Q[0] > Q[2]) goto L5; else goto L6;
    L3: if(Q[0] > Q[2]) return Q[2]; else return Q[0];
    L4: if(Q[3] > Q[1]) return Q[1]; else return Q[3];
    L5: if(Q[0] > Q[3]) return Q[3]; else return Q[0];
    L6: if(Q[2] > Q[1]) return Q[1]; else return Q[2];
}
```

This looks like a full, perfect binary tree. Same cost (4) on all paths. Highly unusual.

Note: the states corresponding to L1 and L2 are isomorphmic, and so are the four last states, L3 to L6.
The notion of state equivalence is key to the calculation.
Computational process, Cont.

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\( d_n \) — number of labeled transitive DAGs, on \( n \) nodes.

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\( x_n \) — number of states needed to find median.

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The largest we did was \( n = 11 \):

\[ d_{11} = 1,396,281,677,105,899; \quad d_{11}^* = 46,749,427. \]
Observations

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<td>5.2242</td>
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<td>5.5255</td>
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<td>310.71</td>
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<td>3471.2</td>
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<td>1297.9</td>
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