On the number of inversions in labelled tree families

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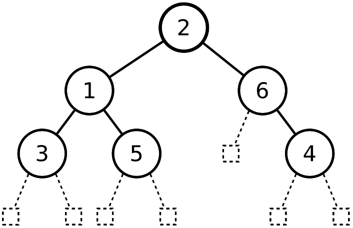
Trees

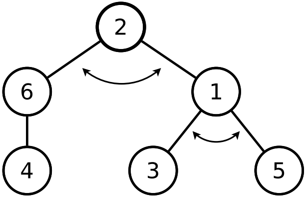
In my talk, all trees are **rooted** and **labelled** with distinct integers (usually: with $\{1, \ldots, |T|\}$, where $|T|$ is the size of tree T).

Examples:

Binary trees



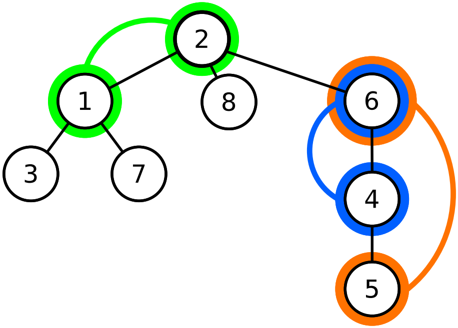
Cayley trees



Inversions

Definition

Let T be a rooted tree labelled with distinct integers.
An **inversion** in T is a pair of nodes (i, j), such that $i > j$ and node i lies on the path from the root to node j.



Existing work on the number of inversions in trees

Mallows, Riordan (1968),
Gessel, Sagan, Yeh (1995)
$\Rightarrow$ Exact enumeration results are hard to find

Flajolet, Poblete, Viola (1998)
$\Rightarrow$ Number of inversions in Cayley trees: asymptotically Airy distributed

Simply generated tree families

- introduced by Meir and Moon (1978)
- include many important tree families (e.g. d-ary trees, Cayley trees, Motzkin trees, cyclic trees, ...)

Each rooted plane tree gets a weight:

$(\varphi_k)_{k\geq 0}$ ... *degree-weight sequence* (nonnegative real numbers, $\varphi_0 > 0$)
$d^+(v)$ ... out-degree (= number of children) of node v

$$w(T) := \prod_{v\in T} \varphi_{d^+(v)}.$$

$\Rightarrow$ Tree family $\mathcal{T}$ defined by $(\varphi_k)_{k\geq 0}$:
Consists of all trees T with $w(T) \neq 0$ together with their weights.

Simply generated tree families

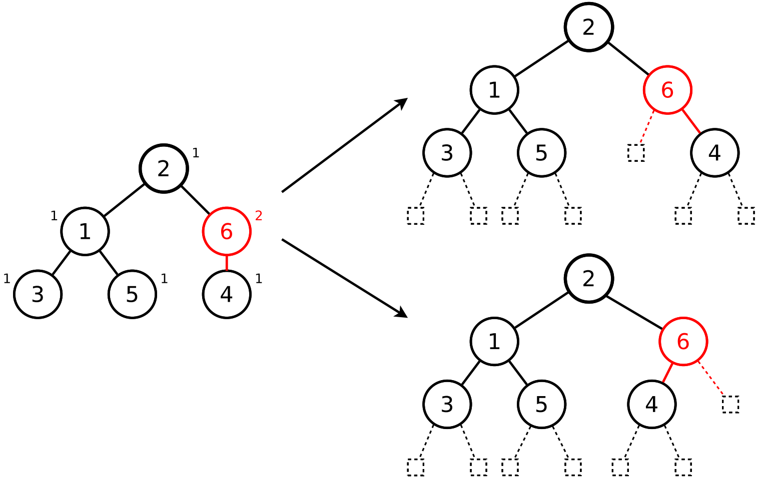
Define the **total weight** of all trees of size n in $\mathcal{T}$:

$$T_n := \sum_{|T|=n} w(T).$$

If all T_n are integers $\Rightarrow$ can be interpreted as the **number** of trees of size n in $\mathcal{T}$.

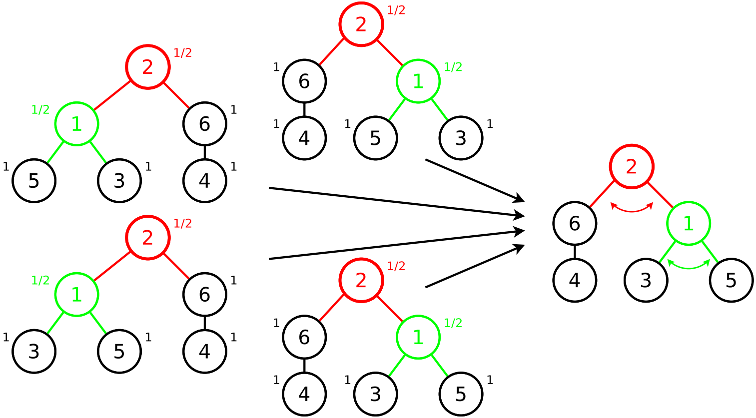
Example

$$(\varphi_k)_{k\geq 0} = (1, 2, 1, 0, 0, 0, \ldots) \quad \Rightarrow \quad \text{Binary trees}$$



Example

$$(\varphi_k)_{k\geq 0} = \left(\frac{1}{k!}\right)_{k\geq 0} \quad \Rightarrow \quad \text{Cayley trees}$$



Simply generated tree families

$T_n := \sum_{|T|=n} w(T)$
$\varphi(t) := \sum_{k\geq 0} \varphi_k t^k$

$\Rightarrow$ **Exponential generating function** of T_n,

$$T(z) := \sum_{n\geq 0} T_n \frac{z^n}{n!},$$

satisifies **functional equation**

$$T(z) = z\varphi(T(z)).$$

Simply generated tree families

formal equation $\mathcal{T} = \circ * \varphi(\mathcal{T}) \quad \Longleftrightarrow \quad T(z) = z\varphi(T(z))$

Binary trees:

$$\mathcal{T} = \circ * (\{\Box\} \,\dot{\cup}\, \mathcal{T}) * (\{\Box\} \,\dot{\cup}\, \mathcal{T}) \quad \Longleftrightarrow \quad T(z) = z(1 + T(z))^2$$

Cayley trees:

$$\begin{aligned} \mathcal{T} &= \circ * \textsc{Set}(\mathcal{T}) \\ &= \circ * (\{\Box\} \,\dot{\cup}\, \mathcal{T} \,\dot{\cup}\, \frac{\mathcal{T}^2}{2!} \,\dot{\cup}\, \frac{\mathcal{T}^3}{3!} \,\dot{\cup}\, \ldots) \quad \Longleftrightarrow \quad T(z) = z\exp(T(z)) \end{aligned}$$

($\dot{\cup}$ ... disjoint union)

Main result I ("global" behaviour)

$\mathcal{T}$ ... labelled family of simply generated trees associated to $(\varphi_k)_{k\geq 0}$
I_n ... number of inversions of a random element of $\mathcal{T}$ of size n
(trees are chosen with probability proportional to their weight)

Technical requirement: $\varphi(t) = \sum_{k\geq 0} \varphi_k t^k$ has positive radius of convergence R and the equation $t\varphi'(t) = \varphi(t)$ has a minimal positive solution $\tau < R$.

"Global" behaviour

$\Rightarrow \mathbb{E}\left(I_n\right) \sim c_\varphi \sqrt{\pi} n^{3/2}$, where $c_\varphi = \sqrt{\frac{\varphi(\tau)}{8\tau^2\varphi''(\tau)}}$, and

$$\frac{I_n}{c_\varphi n^{3/2}} \xrightarrow{(d)} I,$$

where **I is an Airy distributed random variable**.

Airy distribution

The Airy distribution is the probability distribution of a random variable X with support on $[0,\infty)$ which is uniquely determined by its moments,

$$\mathbb{E}(X^r) = -\frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(\frac{3r-1}{2}\right)} C_r,$$

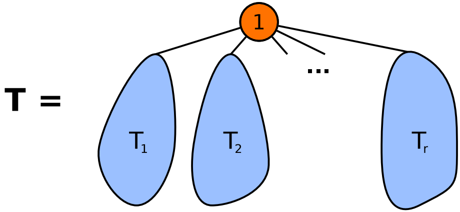
where $C_0 = -1$, and

$$2C_r = (3r-4)C_{r-1} + \sum_{j=1}^{r-1} \binom{r}{j} C_j C_{r-j}, \quad r \geq 1.$$

$\Rightarrow$ we obtain our result using the **method of moments**.

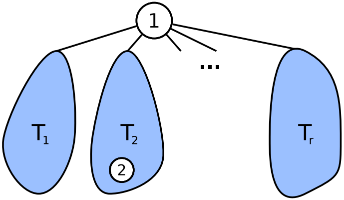
Sketch of proof - Basic ideas

We consider the **tree family** $\hat{\mathcal{T}}$, which contains only the trees of $\mathcal{T}$ with label 1 at the root:



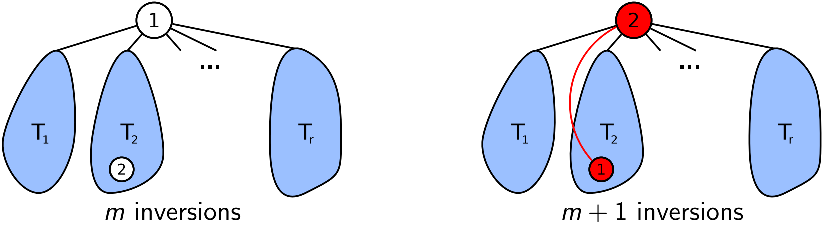
$$inv(T) = inv(T_1) + \ldots + inv(T_r)$$

Sketch of proof - Basic ideas

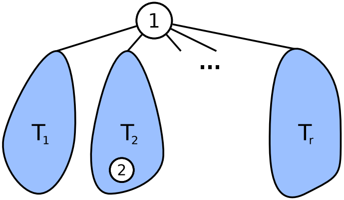


m inversions

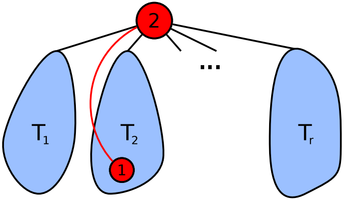
Sketch of proof - Basic ideas



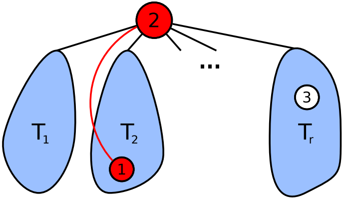
Sketch of proof - Basic ideas



m inversions

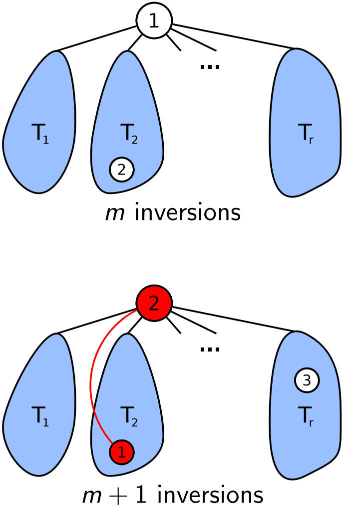


$m + 1$ inversions



$m + 1$ inversions

Sketch of proof - Basic ideas



Sketch of proof - Basic ideas

$\hat{T}_{n,m}$ / $T_{n,m}$ ... total weight of trees of size n with m inversions in $\hat{\mathcal{T}}$ / $\mathcal{T}$

$\Rightarrow$ The generating functions

$$\hat{T}(z,q) := \sum_{n,m\geq 0} \hat{T}_{n,m} \frac{z^n}{n!} q^m,$$

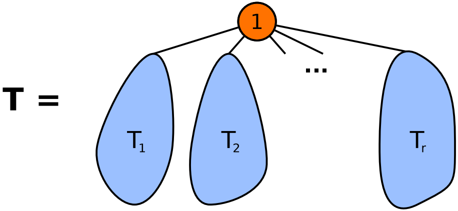
and

$$T(z,q) := \sum_{n,m\geq 0} T_{n,m} \frac{z^n}{n!} q^m,$$

satisfy

$$\begin{aligned} T(z,q) &= \sum_{n,m\geq 0} \underbrace{(1+q+\ldots+q^{n-1})}_{\frac{1-q^n}{1-q}} \hat{T}_{n,m} \frac{z^n}{n!} q^m, \\ &= \frac{\hat{T}(z,q) - \hat{T}(zq,q)}{1-q} =: \mathrm{H}\,\hat{T}(z,q). \end{aligned}$$

Sketch of proof - Basic ideas



formal equation: $\hat{\mathcal{T}} = \textcircled{1} \times \varphi(\mathcal{T}) = \{\bigcirc\}^{\Box} * \varphi(\mathcal{T})$.

$\Rightarrow$ **leads to:**

$$\mathrm{D}_z \hat{T}(z,q) = \varphi(T(z,q)) = \varphi(\mathrm{H}\hat{T}(z,q)).$$

$(\mathrm{D}_z := \frac{\partial}{\partial z})$

Sketch of proof - Basic ideas

$$\mathsf{D}_z \hat{T}(z,q) = \varphi(\mathsf{H}\hat{T}(z,q))$$

Further approach:

- The moments of $\hat{I}_n$ can be computed from $\mathrm{UD}_q^r \hat{T}(z,q)$.
  (U: evaluation at $q = 1$)
  $\Rightarrow$ Apply UD_q^r to the equation for $\hat{T}(z,q)$.
  Problem: equation still contains H!
- Use a trick by Flajolet, Poblete, Viola to get rid of H:

$$\mathrm{UD}_q^j \mathsf{H} = \sum_{s=0}^{j} \binom{j}{s} \frac{1}{s+1} \mathrm{Z}^{s+1} \mathsf{D}_z^{s+1} \mathrm{UD}_q^{j-s}.$$

  $\Rightarrow$ Leads to a recursion for $\mathrm{UD}_z^i \mathsf{D}_q^j \hat{T}(z,q)$.
- Singularity analysis

Main result II ("local behaviour")

$I_{n,j}$ ... number of inversions of the type (i, j) (i.e., $i > j$ and i ancestor of j) in a random element of $\mathcal{T}$ of size n

"Local" behaviour

- Region $n - j \gg \sqrt{n}$: $\frac{\sqrt{n}}{n-j} I_{n,j} \xrightarrow{(d)} X_\sigma$, where X_σ is **Rayleigh distributed** with parameter $\sigma := \sqrt{\frac{\varphi(\tau)}{\tau^2 \varphi''(\tau)}}$.
- Region $n - j \sim \alpha\sqrt{n}$, $\alpha \in \mathbb{R}^+$: $I_{n,j} \xrightarrow{(d)} Y_\gamma$, Y_γ **discrete r.v.**:

$$\mathbb{P}\{Y_\gamma = k\} = \frac{\gamma^k}{k!} \int_0^\infty x^{k+1} e^{-\frac{x^2}{2} - \gamma x} dx, \quad k \in \mathbb{N},$$

$\gamma := \alpha \sqrt{\frac{\varphi(\tau)}{\tau^2 \varphi''(\tau)}}$.

- Region $n - j \ll \sqrt{n}$: $I_{n,j} \xrightarrow{(d)} 0$.

Rayleigh distribution

The Rayleigh distribution with parameter $\sigma > 0$ is the distribution of a random variable X_σ with probability density function

$$f_\sigma(x) = \frac{x}{\sigma^2}\, e^{-\frac{x^2}{2\sigma^2}}, \quad x > 0.$$

It is uniquely determined by its moments,

$$\mathbb{E}\,(X_\sigma^r) = \sigma^r\, 2^{\frac{r}{2}}\, \Gamma\Big(1 + \frac{r}{2}\Big).$$

Sketch of proof - Basic ideas

Construct a **coloured family** $\mathcal{T}_C$ from $\mathcal{T}$: In each tree of $\mathcal{T}_C$

- one node (say, j) is red,
- all nodes with label $< j$ are white,
- all nodes with label $> j$ are black.

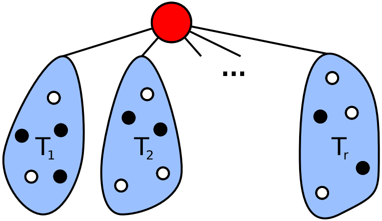
$\Rightarrow$ total weight of trees of size n in $\mathcal{T}$ with k **inversions of type** (i, j)
$=$ total weight of trees of size n in $\mathcal{T}_C$ where $j-1$ **nodes are white** and
k **ancestors of the red node are black**

Generating function:

$$N(z, u, q) = \sum_{T_C \in \mathcal{T}_C} w(T_C) \frac{z^{\sharp \text{ white}}}{(\sharp \text{ white})!} \frac{u^{\sharp \text{ black}}}{(\sharp \text{ black})!} q^{\sharp \text{ black ancestors of red}}.$$

Sketch of proof - Basic ideas

Decompose trees in $\mathcal{T}_C$ with respect to the root:
Case 1: Root node is red: no black ancestors, all subtrees are "bicoloured".

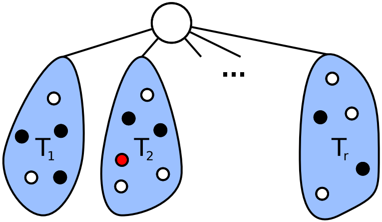


$$\Rightarrow N(z,u,q) = \sum_{\ell \geq 0} \varphi_\ell \big(B(z,u)\big)^\ell + \dots$$

($B(z,u)$: generating function for "bicoloured" trees $\rightarrow$ later)

Sketch of proof - Basic ideas

Decompose trees in $\mathcal{T}_C$ with respect to the root:
Case 2: Root node is white: red node in a subtree T_s, all other subtrees "bicoloured", number of black ancestors in T_C = number of black ancestors in T_s.

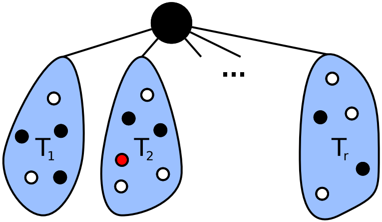


$$\Rightarrow N(z,u,q) = \ldots + z \sum_{\ell \geq 0} \ell \varphi_\ell \big(B(z,u)\big)^{\ell-1} N(z,u,q) + \ldots$$

($B(z,u)$: generating function for "bicoloured" trees $\rightarrow$ later)

Sketch of proof - Basic ideas

Decompose trees in $\mathcal{T}_C$ with respect to the root:
Case 3: Root node is black: red node in a subtree T_s, all other subtrees "bicoloured", number of black ancestors in $T_C = 1 +$ number of black ancestors in T_s.



$$\Rightarrow N(z,u,q) = \ldots + uq \sum_{\ell \geq 0} \ell \varphi_\ell \big(B(z,u)\big)^{\ell-1} N(z,u,q).$$

($B(z,u)$: generating function for "bicoloured" trees $\to$ later)

Sketch of proof - Basic ideas

Easy: $B(z,u) = T(z+u)$

$$\Rightarrow N(z,u,q) = \frac{\varphi(T(z+u))}{1-(z+uq)\varphi'(T(z+u))}.$$

Further approach:

- Apply UD_q^r to $N(z,u,q) \Rightarrow$ generating function of factorial moments
- Singularity analysis
- Method of moments

Thank you for your attention!