Counting increasing chordal graphs

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SORBONNE UNIVERSITÉS

Outline

- Increasing chordal graphs definition.
- Some useful subclasses.
- A bijection between increasing chordal graphs & a family of increasing trees.
- Applications of the bijection.

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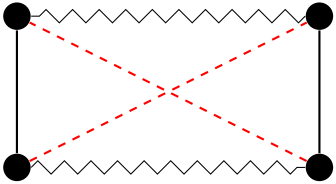
- Some useful subclasses.
- A bijection between increasing chordal graphs & a family of increasing trees.
- Applications of the bijection. ¡Work in progess!

What is a chordal graph?

Definition 1

G is chordal if
every cycle of length ≥ 4 in G
has a chord.

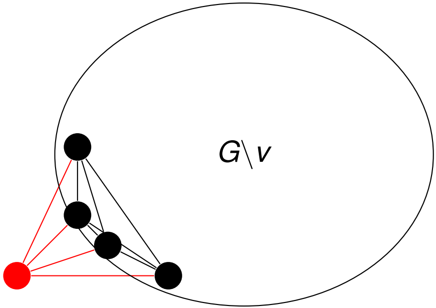


What is a chordal graph?

Definition 2 [Fulkerson Gross, 1965]

G is chordal if
it has a vertex v whose
neighborhood is a clique &
$G \setminus v$ is chordal.

(Perfect elimination ordering)



What is an increasing chordal graph?

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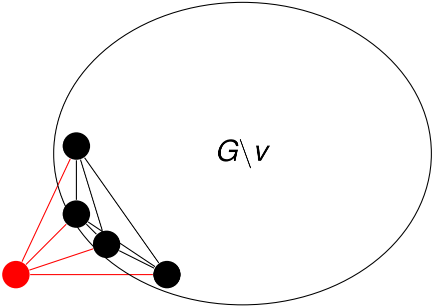


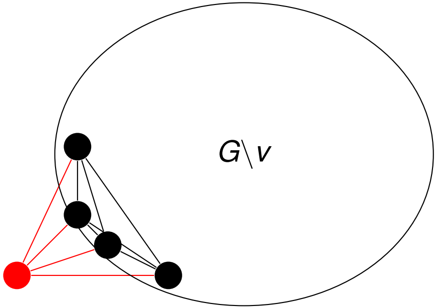
Increasing $\equiv$ Labelling compatible with elimination.

Red node has largest label.

Subclass: k-trees

Definition

G is chordal if
it has a vertex v whose neighborhood is a clique &
$G \backslash v$ is chordal &
the clique is of size k.



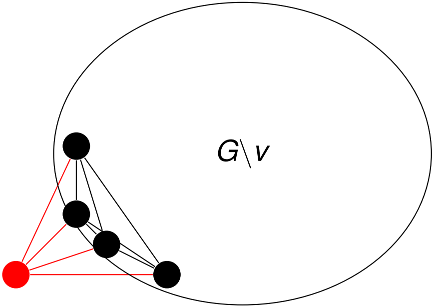
Increasing $\equiv$ Labelling compatible with elimination.

Red node has largest label.

Subclass: planar k-trees

Definition

G is chordal if
it has a vertex v whose neighborhood is a clique &
$G \setminus v$ is chordal &
the clique is of size k &
every k-clique has only one or two neighbors.



planar 2-trees $\equiv$ polygon triangulations
planar 3-trees $\equiv$ Apollonian networks, stack triangulations

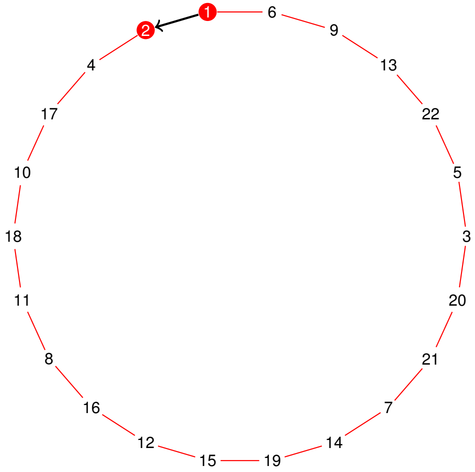
Increasing $\equiv$ Labelling compatible with elimination.
Red node has largest label.

Bijection with a family of increasing trees

Remark

The perfect elimination schema implies a tree-like structure (tree decomposition).

Triangulations of a regular polygon ↔ binary trees



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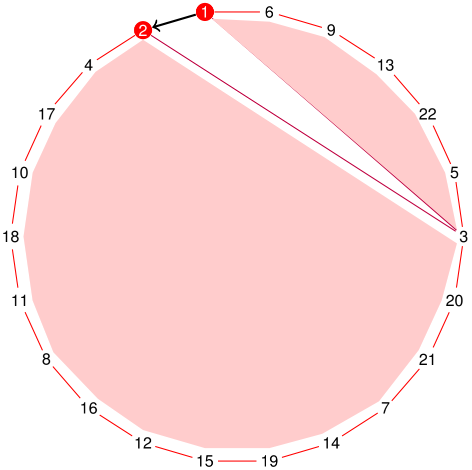

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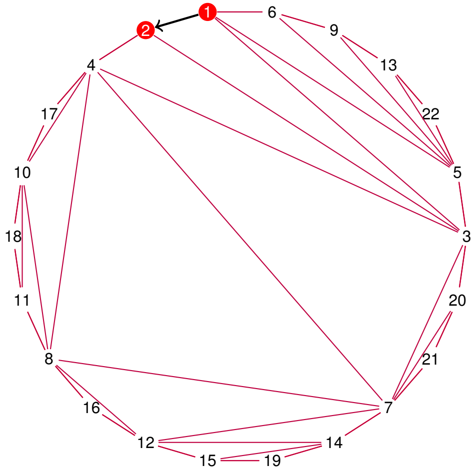
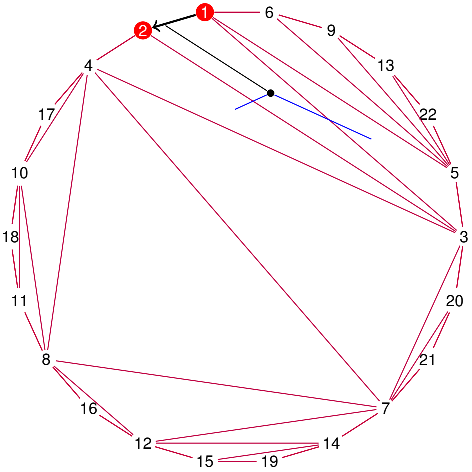
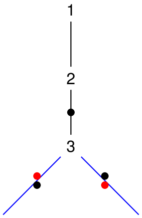


Triangulations of a regular polygon ↔ binary trees





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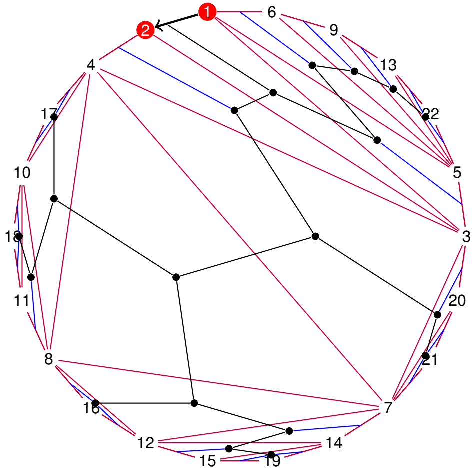
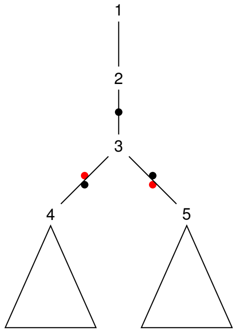
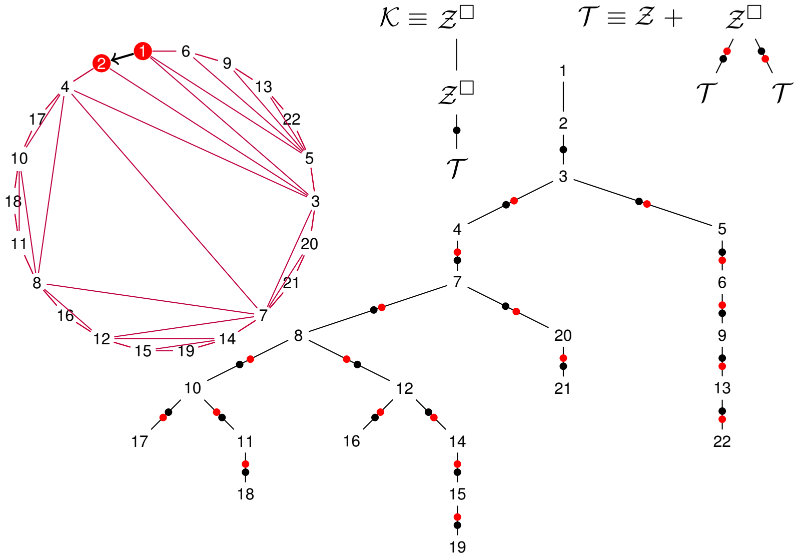
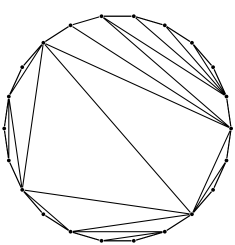


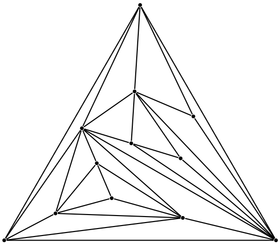


Triangulations of a regular polygon ↔ binary trees

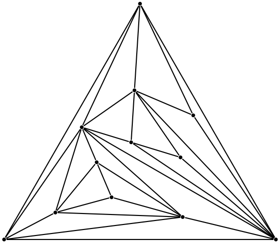


Generalization to planar k-trees

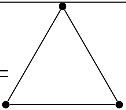
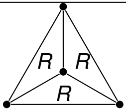




$k = 2$	$k = 3$	any k
binary trees	ternary trees	k-ary trees

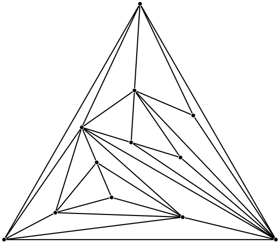


planar

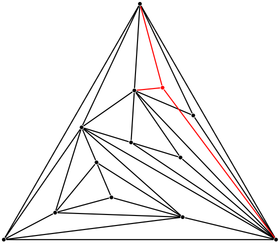
$R =$  or 

$$\mathcal{T} = \mathcal{Z} + \mathcal{Z}^{\square} \star \mathcal{T}^{k}$$

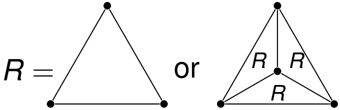
Generalization to (non-planar) k-trees



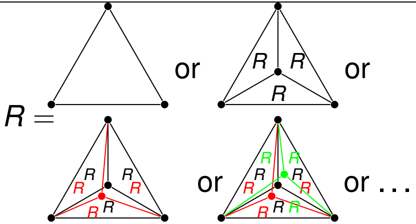
planar



general



$$\mathcal{T} = \mathcal{Z} + \mathcal{Z}^{\square} \star \mathcal{T}^{k}$$

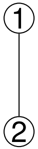


$$\mathcal{T} = \mathcal{Z}^{\square} \star \textsc{Set}(\mathcal{T})^{k}$$

The bijection for chordal graphs

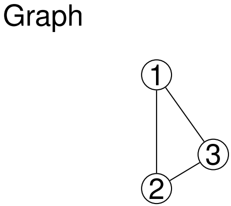
The bijection for chordal graphs

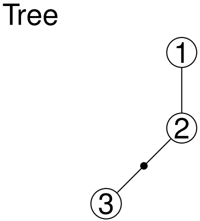
Graph



Tree

The bijection for chordal graphs





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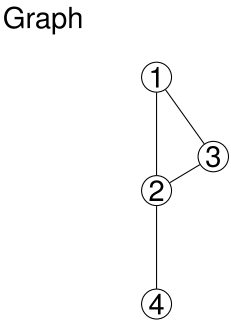

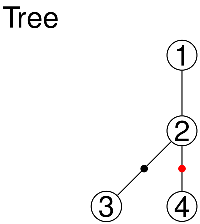

The bijection for chordal graphs

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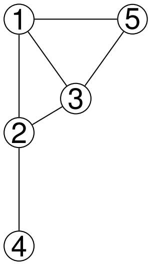
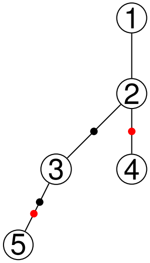


Tree



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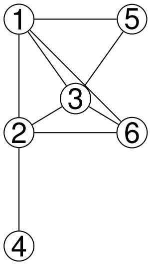

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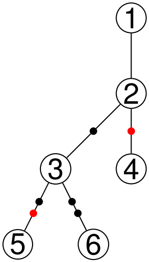

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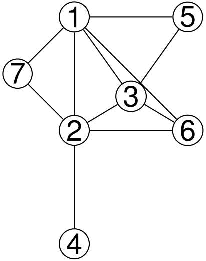

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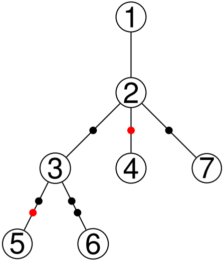

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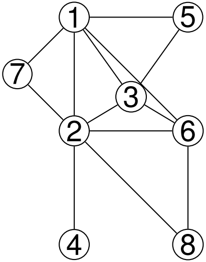

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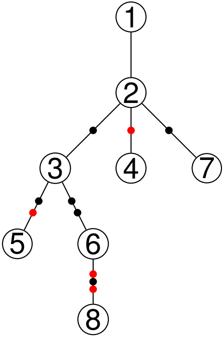


Counting

k counts the size of the clique above the tree's root

$$\mathcal{T}_k = \mathcal{Z}^{\square} \star \textsc{Set}\left(\sum_{k'=1}^{k+1} \binom{k}{k'-1} \mathcal{T}_{k'}\right)$$

$$T_k(z) = \int_0^z \exp\left(\sum_{k'=1}^{k+1} \binom{k}{k'-1} T_{k'}(x)\right) \mathrm{d}x$$

1
1
3
17
153
2005
36435
892589
29024097
1243719721
70126736211
5214340717481
513580908313113
67385243248199869
11849902742218296387
2809988385697488803525
903790557022400488011393
396433825475581410279880273
238335152020259980288481504355
197287549002583341310761895590593

Distances to vertex 1

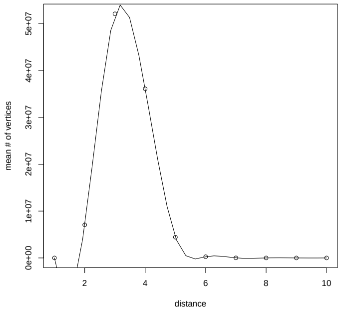
k counts the size of the clique above the tree's root

d counts the distance of the vertex in the tree's root to vertex 1

i counts the number of vertices in the clique above the tree's root that are at distance d from vertex 1, $0 \leq i \leq k$

$$\mathcal{T}_{k,d,i} = \mathcal{Z}_d^{\square} \star \textsc{Set}\left(\sum_{k'=1}^{k+1}\sum_{i'=0}^{k'-1}\binom{k-i}{k'-i'}\binom{i}{i'-1}\mathcal{T}_{k',d,i'}\right)$$

Profile of a chordal graph of size 10^8:



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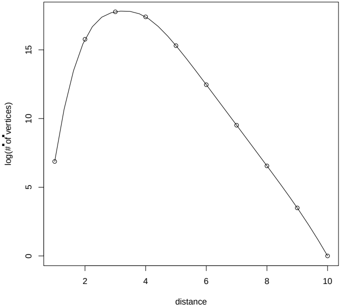
Log of Profile of a chordal graph of size 10^8:



What next?

- Coefficient asymptotics (bounded and unbounded system)
- Profile calculation
- Degree distribution
- Counting labeled chordal graph (see also [Wormald 1985])
- ...