Boltzmann generation for languages with Shuffle product

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Joint work with A. Darasse, K. Panagiotou and M. Soria

Shuffle product

Shuffle product of two words $u, v \in \Sigma^*$

$$u ⧢ v = \{u_1 v_1 u_2 v_2 \cdots u_n v_n \quad | \quad u = u_1 \cdots u_n \ , \quad v = v_1 \cdots v_n\}$$

Example

$$abc ⧢ de = \{abcde, abdce, adbce, dabce, abdec,$$
$$adbec, dabec, adebc, daebc, deabc\}$$

Shuffle product of two languages $\mathcal{A}$ and $\mathcal{B}$

$$\mathcal{C} = \mathcal{A} ⧢ \mathcal{B} = \{u ⧢ v \quad | \quad u \in \mathcal{A} \ , \quad v \in \mathcal{B}\}$$

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$$

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$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$$

Generating functions

Symbolic method

- If $\mathcal{C}$ is a language, we associate two generating functions
- Formal power series
- Let $c_n = |\mathcal{C}_n| = |\{w \in \mathcal{C} \quad | \quad |w| = n\}|$. We define

$$OGF(\mathcal{C})(z) = C(z) = \sum_{n=0}^{\infty} c_n z^n \quad ; \quad EGF(\mathcal{C})(z) = \widehat{C}(z) = \sum_{n=0}^{\infty} c_n \frac{z^n}{n!}$$

- Same enumeration sequence c_n

Example

Let $\mathcal{C} = \{a^n \quad | \quad n \in \mathbb{N}\} = \{\epsilon, a, aa, aaa, \cdots\}$. Then

$$C(z) = \frac{1}{1-z} \qquad \text{and} \qquad \widehat{C}(z) = e^z.$$

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Uniform random generation – Boltzmann model

[Ordinary | Exponential] Boltzmann sampler for a language $\mathcal{C}$

- An algorithm, named $[\Gamma\mathcal{C} \mid \widehat{\Gamma}\mathcal{C}]$
- Depends on a real parameter x
- Returns a word $\gamma \in \mathcal{C}$ with probability

$$\mathbb{P}_x(\gamma) = \frac{1}{C(z)} x^{|\gamma|} \qquad \mathbb{P}_x(\gamma) = \frac{1}{\widehat{C}(z)} \frac{x^{|\gamma|}}{|\gamma|!}$$

Properties

- Uniformity in each size
- Simple probabilistic algorithms: simple derivation from specifications
- Efficiency: free sampler of complexity essentially linear in the size of the output

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Examples of ordinary Boltzmann samplers

Boltzmann samplers $\Gamma\mathcal{C}(x)$

If $\mathcal{C} = \mathcal{A} + \mathcal{B}$
$\longrightarrow C(z) = A(z) + B(z)$

1: **if** *Bernoulli* $\left(\frac{A(x)}{C(x)}\right)$ **then**
2: **return** $\Gamma\mathcal{A}(x)$
3: **else**
4: **return** $\Gamma\mathcal{B}(x)$
5: **end if**

If $\mathcal{C} = \mathcal{S}eq(\mathcal{A}) = \varepsilon + \mathcal{A} \times \mathcal{C}$
$\longrightarrow C(z) = \frac{1}{1-A(z)}$

1: **if** *Bernoulli* $\left(\frac{1}{C(x)}\right)$ **then**
2: **return** ε
3: **else**
4: **return** $(\Gamma\mathcal{A}(x), \Gamma\mathcal{C}(x))$
5: **end if**

If $\mathcal{C} = \mathcal{A} \times \mathcal{B}$ $\longrightarrow C(z) = A(z) \times B(z)$

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Independent calls!

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```
if Bernoulli(A(x)/C(x)) then
    return ΓA(x)
else
    return ΓB(x)
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Independent calls!

Our goal

Regular specification with Shuffle

$$\mathcal{L} = (a + b^* c)^* \sqcup\!\sqcup (ba^* \sqcup\!\sqcup (dc)^*)$$

Iterative specification, involving *atoms*, the operators *Union*, *Product* and *Sequence*, and the *Shuffle*.

We have Boltzmann samplers for $+$, $\times$ and $\mathcal{S}eq$.

We need one for the *Shuffle product* $\sqcup\!\sqcup$.

Precisely:

Being given $\Gamma\mathcal{A}$ and $\Gamma\mathcal{B}$, if $\mathcal{C} = \mathcal{A} \sqcup\!\sqcup \mathcal{B}$, generate $\gamma \in \mathcal{C}$ such that

$$\mathbb{P}_x(\gamma) = \frac{1}{OGF_{\mathcal{C}}(x)} x^{|\gamma|}$$

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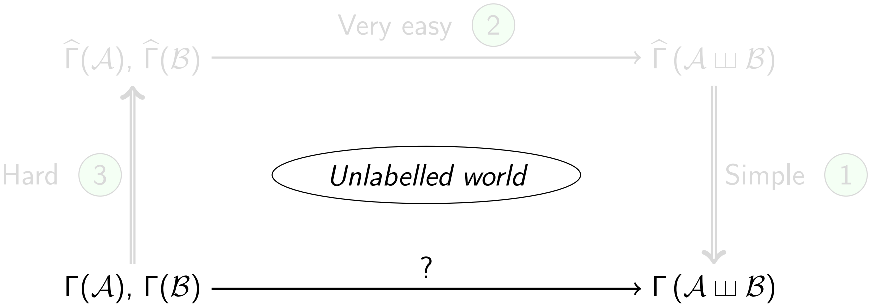

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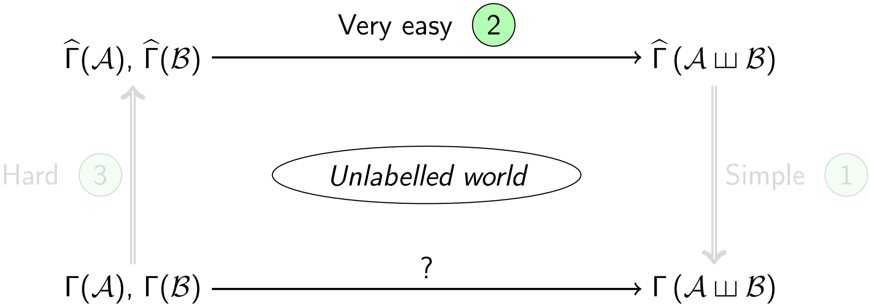

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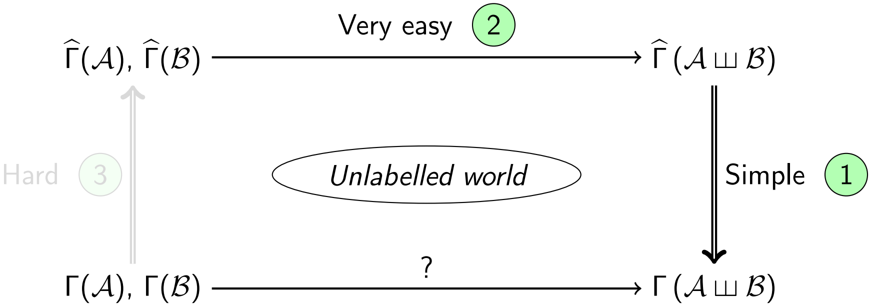

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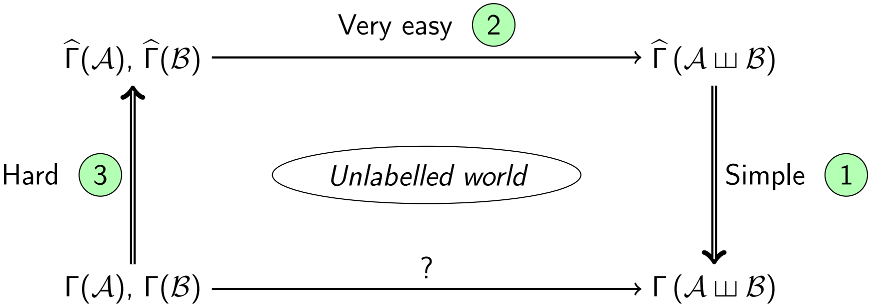

1 ⇓) Transformation $\widehat{\Gamma}\mathcal{C} \to \Gamma\mathcal{C}$

Define the following probability density function on $[0,1]$:

$$\delta_{\mathcal{A},x}(u) = \frac{\widehat{A}(ux)e^{-u}}{A(x)}$$

Algorithm $\Gamma\mathcal{A}(x)$ (1)

1: Draw $u \in [0,1]$ following the $\delta_{\mathcal{A},x}$ density
2: **return** $\widehat{\Gamma}\mathcal{A}(ux)$

Proof

Just compute straight forward the probability of returning a given object $\alpha \in \mathcal{A}$, use the fact that $\widehat{\Gamma}\mathcal{A}$ is an exponential Boltzmann sampler, play with integrals, and *voilà*.

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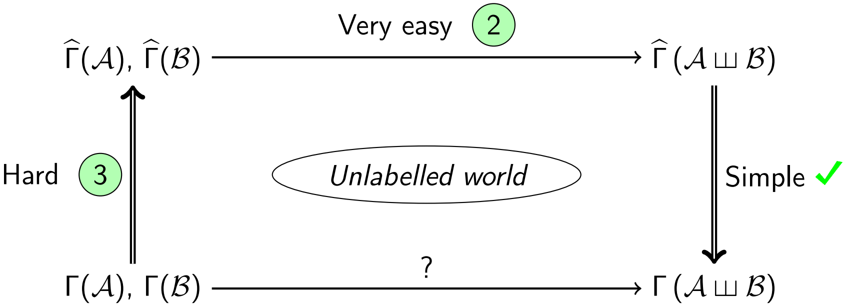
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1bis ⇑) Inverse transformation $\Gamma\mathcal{C} \to \widehat{\Gamma}\mathcal{C}$

Idea

Imagine the same trick for the other way, from ordinary to exponential.

But:

There is no probability density function δ such that $\int u^n \delta(u)\, du = \frac{1}{n!}$.

Impossible moment problem

- Restriction to regular languages (with shuffle)
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2 →) Hat classes

- We need to see our unlabeled words as (uniquely) labeled
- Unlabeled language $\mathcal{A}$ → Uniquely labeled language $\widehat{\mathcal{A}}$
- Canonical embedding: $T : aabba \mapsto a^1 a^2 b^3 b^4 a^5$
- $EGF(\mathcal{A}) = EGF(\widehat{\mathcal{A}})$ and $\widehat{\Gamma}\mathcal{A} = \widehat{\Gamma}\widehat{\mathcal{A}}$

Nice behaviour with respect to shuffle

$$\mathcal{A} ⧢ \mathcal{B} \mapsto \widehat{\mathcal{A} ⧢ \mathcal{B}} = \widehat{\mathcal{A}} \star \widehat{\mathcal{B}}$$

Labelled product = Shuffle of labels

$$EGF(\mathcal{A} ⧢ \mathcal{B}) = EGF(\widehat{\mathcal{A}}) \times EGF(\widehat{\mathcal{B}})$$

Boltzmann sampler (2)

$$\widehat{\Gamma}(\mathcal{A} ⧢ \mathcal{B}) = \left(\widehat{\Gamma}(\mathcal{A}), \widehat{\Gamma}(\mathcal{B})\right)$$

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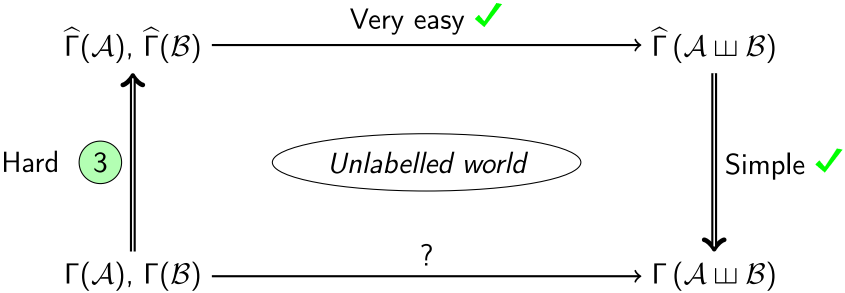

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3 $\Uparrow$) Hat classes: Union, Product, Sequence

Union $\rightarrow$ Union

- $\widehat{\mathcal{A}+\mathcal{B}} = \widehat{\mathcal{A}} + \widehat{\mathcal{B}}$
- $EGF(\widehat{\mathcal{A}+\mathcal{B}}) = EGF(\widehat{\mathcal{A}}) + EGF(\widehat{\mathcal{B}})$
- $\widehat{\Gamma}(\widehat{\mathcal{A}+\mathcal{B}}) =$ Bernoulli, then $\widehat{\Gamma}(\widehat{\mathcal{A}})$ or $\widehat{\Gamma}(\widehat{\mathcal{B}})$

Sequence $\rightarrow$ Product and Union

- $\mathcal{C} = \mathcal{S}eq(\mathcal{A}) \longrightarrow \mathcal{C} = \varepsilon + \mathcal{A} \times \mathcal{C}$
- $\Rightarrow \widehat{\mathcal{C}} = \varepsilon + \widehat{\mathcal{A} \times \mathcal{C}}$

Cartesian product $\rightarrow$ Ordered product

- $\widehat{\mathcal{A} \times \mathcal{B}} = \widehat{\mathcal{A}} \odot \widehat{\mathcal{B}}$
- $EGF(\widehat{\mathcal{A} \times \mathcal{B}}) = ????$
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Ordered product

Example

If $\widehat{\alpha} = a^1 b^2 b^3 a^4$ and $\widehat{\beta} = a^1 c^2 a^3$, then

$$\widehat{\alpha} \odot \widehat{\beta} = a^1 b^2 b^3 a^4 a^5 c^6 a^7$$

Definition

$$\widehat{\mathcal{A}} \odot \widehat{\mathcal{B}} = \{\widehat{\alpha} \cdot \widehat{\beta} \quad | \quad \widehat{\alpha} \in \widehat{\mathcal{A}}\,, \quad \widehat{\beta} \in \widehat{\mathcal{B}}\,, \quad \beta\text{'s labels translated by } |\widehat{\alpha}|\}$$

Property

$$\widehat{\mathcal{A}} \odot \widehat{\mathcal{B}} = \mathcal{A}_0 \widehat{\mathcal{B}} + \widehat{\mathcal{A}_{\geqslant 1}} \odot \widehat{\mathcal{B}}$$

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Derivative of labeled languages

Example

Let $\widehat{\alpha} = a^1 b^2 c^3$ with $|\widehat{\alpha}| = 3$. Then $\widehat{\alpha}' = (a) b^1 c^2$ with $|\widehat{\alpha}'| = 2$, and $\widehat{\alpha}'' = (a)(b) c^1$ with $|\widehat{\alpha}''| = 1$.

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We define $\widehat{\mathcal{A}}' = \{\widehat{\alpha}' \quad | \quad \widehat{\alpha} \in \widehat{\mathcal{A}}$ and $|\widehat{\alpha}| = n > 0$, where we replace the smallest letter by a *hole* and relabel from 1 to $n-1\}$.

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Exponential generating function for ordered product

Property of ordered product

$$\widehat{\mathcal{A}} \odot \widehat{\mathcal{B}} = \mathcal{A}_0 \widehat{\mathcal{B}} + \widehat{\mathcal{A}_{\geqslant 1}} \odot \widehat{\mathcal{B}} \cong \mathcal{A}_0 \widehat{\mathcal{B}} + \widehat{\mathcal{A}'} \odot \mathcal{Z} \odot \widehat{\mathcal{B}}$$

- $\widehat{\alpha} \cdot \widehat{\beta} = \widehat{\beta}$ if $\widehat{\alpha} = \varepsilon$
- $\widehat{\mathcal{A}_{\geqslant 1}} \cong \mathcal{Z} \odot \widehat{\mathcal{A}'}$:
  $\widehat{\alpha} = a^1 b^2 c^3 \to \widehat{\alpha}' = (a) b^1 c^2 \to \mathcal{Z} \odot \widehat{\alpha}' = {Z_a}^1 b^2 c^3 \cong a^1 b^2 c^3$

EGF of ordered product

If $\widehat{\mathcal{C}} = \widehat{\mathcal{A}} \odot \widehat{\mathcal{B}}$, then

$$\widehat{C}(z) = a_0 \widehat{B}(z) + \int_0^z \widehat{A}'(z-t)\widehat{B}(t)\,dt$$

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Boltzmann samplers for derivatives of classes

Problem

- $\widehat{\mathcal{A}} \odot \widehat{\mathcal{B}} \longrightarrow$ derivative class $\widehat{\mathcal{A}'}$
- $\widehat{\Gamma}(\widehat{\mathcal{A}} \odot \widehat{\mathcal{B}}) \longrightarrow$ derivative Boltzmann sampler $\widehat{\Gamma}(\widehat{\mathcal{A}}') = \widehat{\Gamma}^{(1)}(\widehat{\mathcal{A}})$
- Recursively, we need $\widehat{\Gamma}^{(k)}(\widehat{\mathcal{A}})$

***k*-th derivative behaves nicely**

$$\left(\widehat{\mathcal{A}+\mathcal{B}}\right)^{(k)} = \widehat{\mathcal{A}}^{(k)} + \widehat{\mathcal{B}}^{(k)} \qquad \left(\widehat{\mathcal{A} \sqcup \mathcal{B}}\right)^{(k)} = \left(\widehat{\mathcal{A}} \star \widehat{\mathcal{B}}\right)^{(k)}$$

$$\left(\widehat{\mathcal{A}\times\mathcal{B}}\right)^{(k)} = \left(\widehat{\mathcal{A}} \odot \widehat{\mathcal{B}}\right)^{(k)}$$

We have designed Boltzmann samplers for the *k*-th derivatives of languages

$\Longrightarrow$ We have a Boltzmann sampler for the ordered product $\widehat{\Gamma}(\widehat{\mathcal{A}} \odot \widehat{\mathcal{B}})$

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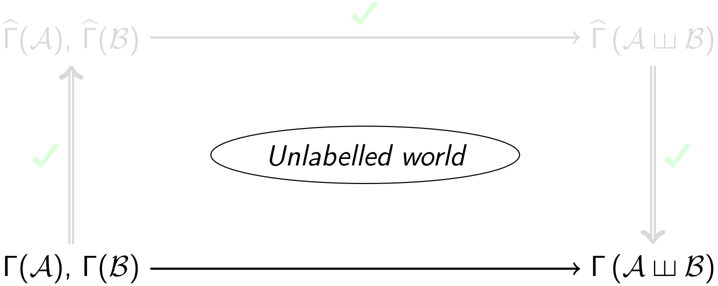

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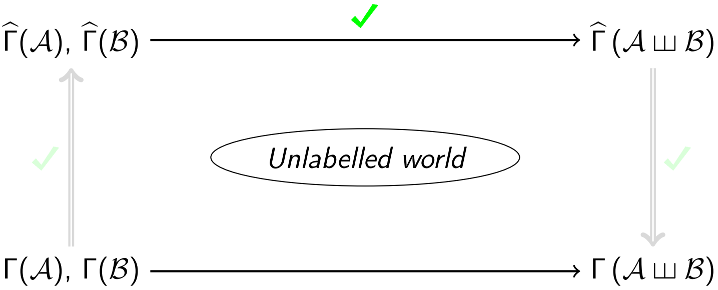

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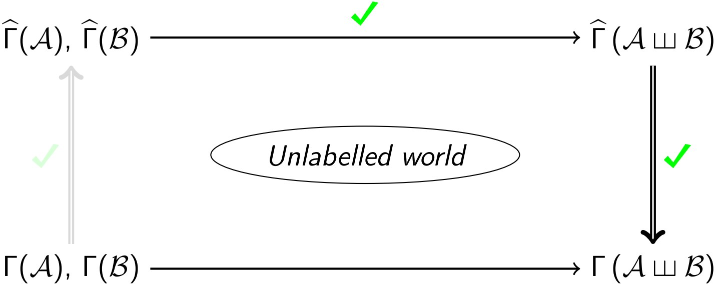

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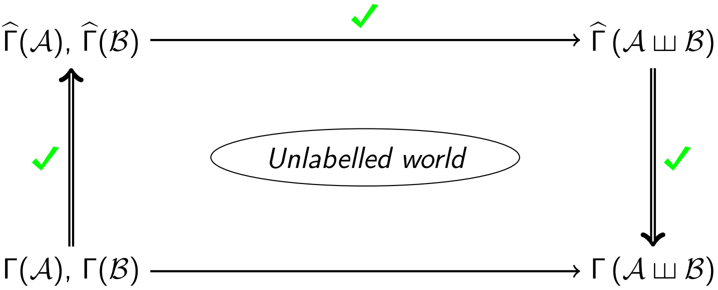

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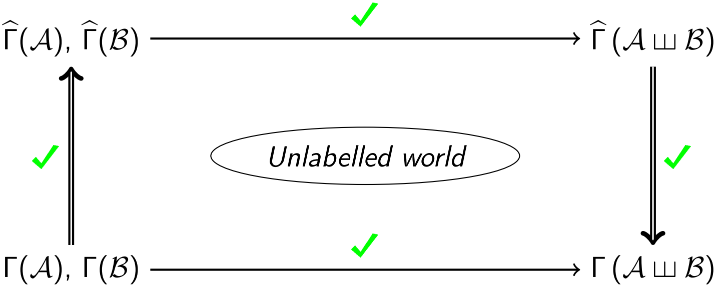

Conclusion

Theorem [DPRS10]

Let $\mathcal{C}$ be a regular language with shuffle. Then there exists an ordinary Boltzmann sampler $\Gamma\mathcal{C}(x)$ linear in the size of the output. And we can automatically build it from the specification of $\mathcal{C}$.

Perspectives and work in progress

- Extension to linear languages with shuffle (done: e.g. $\{a^n b^n | n \in \mathbb{N}\}$)
- Extension to closure by substitution of linear languages with shuffle (almost done: e.g. $\{(ab^*)^n (c^m d^m)^n | n, m \in \mathbb{N}\}$)

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